Localization operator in one dimension and its eigenvalues

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Let A, B be two measurable subsets of the real line. We can associate with them the time-frequency localization operator $S_{A,B}$ given by

$$S_{A,B} = P_A \mathcal{F}^{-1} P_B \mathcal{F} P_A,$$

where P_X is a multiplication operator by $\mathbb{1}_X$ and $\mathcal{F}$ is the Fourier transform. It can be shown that $S_{A,B}$ is a self-adjoint non-negative definite operator. Moreover, if A and B have finite measure then $S_{A,B}$ is a Hilbert–Schmidt operator, in particular it is compact. Therefore, it has a sequence of eigenvalues

$$1 > \lambda_1(A, B) \geq \lambda_2(A, B) \geq \ldots \geq 0.$$

In this talk we will discuss the distribution of these eigenvalues in the case when both A and B are intervals. In this case the eigenvalues depend only on the product of lengths of these intervals $c = |A||B|$ and exhibit a phase transition: first $\approx c$ of the eigenvalues are very close to 1, then there are only $\approx \log c$ intermediate eigenvalues, and the remaining eigenvalues decay to 0 very fast.

We will survey known results about the precise behaviour of these eigenvalues in these three regions, and state recent qualitatively sharp estimates for $\lambda_n(A, B)$ when $n < c$. If time permits, we will also mention what is known in higher dimensions and for more general sets A and B.

The talk is based on a joint work with Fedor Nazarov.