The main Γ-convergence result for the cell problem states as follows: 26-11.2024

<u>Th</u>: Fix $1<p<+\infty$ and $f : \mathbb{R}\times\mathbb{R} \longrightarrow [0,+\infty]$ satisfying

i) f is a Carathéodory function;

ii) $\forall\, \xi\in\mathbb{R}$ the map $f(\cdot,\xi)$ is 1-periodic;

iii) for a.e. $x\in\mathbb{R}$ the map $f(x,\cdot)$ is convex;

iv) there exists $c\in\mathbb{R}^+$ s.t. $|\xi|^p \le f(x,\xi) \le c(|\xi|^p+1)$,

and define the sequence $F_\varepsilon : L^p(0,1) \longrightarrow [0,+\infty] \qquad (\varepsilon\in\mathbb{R}^+)$

$$u \longmapsto \begin{cases} \int_0^1 f\left(\frac{x}{\varepsilon}, u'(x)\right)dx & \text{, if } u\in W^{1,p}(0,1) \\ +\infty & \text{, otherwise} \end{cases}.$$

Then, there exists $f_{hom} : \mathbb{R} \longrightarrow [0,+\infty)$ convex s.t. $\{F_\varepsilon\}_\varepsilon$ Γ-converges to

$$F_{hom} : L^p(0,1) \longrightarrow [0,+\infty]$$

$$u \longmapsto \begin{cases} \int_0^1 f_{hom}(u'(x))\,dx & \text{, if } u\in W^{1,p}(0,1) \\ +\infty & \text{, otherwise} \end{cases}$$

in the strong topology of $L^p(0,1)$. Moreover,

$$f_{hom}(\xi) = \min\left\{ \int_0^1 f(x, \xi + u'(x))\,dx : u\in W^{1,p}(0,1) \text{ and } u(0)=u(1) \right\}.$$

<u>Remark</u>: · Note that, by the direct methods in Calc. Var., the previous minimization problem has always solutions.

· We have already showed that $f_{hom}(\xi) = \lim_{k\nearrow+\infty} M_k(\xi)$. It also holds that

$$f_{hom}(\xi) \doteq \lim_{T\nearrow+\infty} g_T(\xi),$$

where

$$g_T(\xi) = \min\left\{ \frac{1}{T}\int_0^T f(x, \xi+u'(x))\,dx : u\in W_0^{1,p}(0,T) \right\},$$

i.e., it holds the asymptotic homogenization formula.

In this 1-dimensional case, one can use a "direct approach" of Γ-convergence, which is based on understanding at first who can be a candidate for the Γ-limit and then show Γ-convergence, by definition.

Unfortunately, it will not be possible in the n-dimensional case ($n>1$) where once needs to use an "indirect approach".

· Before proving the previous theorem, we need the following preliminary result.

<u>Lemma</u>: Let $u_j, u \in W^{1,p}(0,1)$, $j \in \mathbb{N}$, satisfy $u_j \longrightarrow u$ in $L^p(0,1)$ and let $\{\varepsilon_j\}_j \subseteq \mathbb{R}^+$ be such that $\varepsilon_j \searrow 0$ as $j \nearrow +\infty$.

Then, there exists a sequence $\{v_j\}_j \subseteq W^{1,p}(0,1)$ s.t.

a) $v_j - u \in W_0^{1,p}(0,1) \quad \forall j \in \mathbb{N}$

b) $v_j \longrightarrow u$ in $L^p(0,1)$

c) $\displaystyle \limsup_{j \nearrow +\infty} \int_0^1 f\left(\frac{x}{\varepsilon_j}, v_j'(x)\right) dx \leq \limsup_{j \to +\infty} \int_0^1 f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right) dx.$

where f, p satisfy the hypotheses of the previous theorem.

<u>Proof</u>: Fix $\varphi \in W_0^{1,p}(0,1)$ and assume that $\varphi > 0$ in $(0,1)$. Then, let

$$v_j \doteq u + \big(((u_j - u) \wedge \varphi) \vee (-\varphi) \big) \text{ (truncation) } \forall j \in \mathbb{N},$$

where $\begin{cases} f \wedge g \doteq \min\{f, g\} \\ f \vee g \doteq \max\{f, g\} \end{cases}$.

Note that $v_j - u = \begin{cases} u_j - u & , \text{ if } |u_j - u| < \varphi \\ \pm \varphi & , \text{ if } |u_j - u| \geq \varphi \end{cases} \quad \left(\begin{matrix} - \text{ if } u_j - u \leq -\varphi \\ + \text{ if } u_j - u \geq +\varphi \end{matrix} \right)$

and, in both cases, $v_j - u \in W_0^{1,p}(0,1)$ $\forall j \in \mathbb{N}$, and (a) follows.

In particular, when $v_j - u = u_j - u$, it holds by construction that

$$\begin{cases} 0 \leq |u_j - u| < \varphi \\ \varphi \in W_0^{1,p}(0,1) \Rightarrow \overline{\operatorname{supp} \varphi} \subseteq (0,1). \end{cases}$$

Also (b) is satisfied. Indeed, by construction $\forall p \in [1,+\infty)$

$\|v_j - u\|_{L^p}$ behaves like $\|u_j - u\|_{L^p}$ for j big enough.

We conclude by showing the validity of (c).

· Denote $E_j(x) \doteq \{x \in (0,1) : v_j(x) \neq u_j(x)\}$ $\forall j \in \mathbb{N}$ and notice that, by construction, $|E_j| \longrightarrow 0$ as $j \longrightarrow +\infty$. Then

$$\int_0^1 f\left(\frac{x}{\varepsilon_j}, v_j'(x)\right) dx = \int_{(0,1)\setminus E_j} f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right) dx + \int_{E_j} f\left(\frac{x}{\varepsilon_j}, v_j'(x)\right) dx$$

$\begin{cases} 1^\circ: (0,1)\setminus E_j \subseteq (0,1) \text{ and } f \geq 0 \\ 2^\circ: f \leq c(1+|\xi|^p) \end{cases}$

$$\leq \int_0^1 f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right) dx + \int_{E_j} c\left(1+|v_j'|^p\right) dx$$

$\left\{ \text{in } E_j \;\; v_j - u = \pm\varphi \;\; + \text{triang.} \right.$

$$\leq \int_0^1 f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right) dx + \int_{E_j} c\left(1+|u'|^p + |\varphi'|^p\right) dx.$$

Then, the thesis follows being $1+|u'|^p+|\varphi'|^p \in L^1(0,1)$ and $|E_j| \to 0$ as $j \to +\infty$.

$\square$

We are now in the position to prove the previous theorem.

<u>Proof</u>: (1-d periodic homogenization)

Fix a sequence of indices $\{\varepsilon_j\}_j \subseteq \mathbb{R}^+$, $j \in \mathbb{N}$, s.t. $\lim_{j \nearrow +\infty} \varepsilon_j = 0$. To directly prove Γ-convergence, we show the validity of the liminf and limsup inequalities.

1) Γ-limsup inequality: $\forall\, u \in L^p(0,1)$ $\exists\, \bar{u}_j \longrightarrow u$ (strongly) in $L^p(0,1)$ s.t.

Approximation by piecewise affine functions

$$\limsup_{j \nearrow +\infty} F_{\varepsilon_j}(\bar{u}_j) \leq F_{hom}(u) \text{ or, equivalently,}$$

$$\boxed{\left(\Gamma\text{-}\limsup_{j \nearrow +\infty} F_{\varepsilon_j}\right)(u) \leq F_{hom}(u) \quad \forall\, u \in L^p(0,1).}$$

· Fix $u \in W^{1,p}(0,1)$ (we are allowed to do so being both F_ε and $F_{hom} = +\infty$ in $L^p(0,1) \setminus W^{1,p}(0,1)$).

STEP 1: Assume that u is (piecewise) affine, i.e. if we partition the interval $[0,1]$ as $[0,1]=\bigcup_{k=1}^{n}[a_{k-1},a_k]$, with $a_0=0$, $a_n=1$ and $a_{k-1}<a_k\ \forall k$, then

$$\boxed{u(x)=m_k x+q_k} \qquad \forall\, x\in[a_{k-1},a_k],\ \text{with } m_k, q_k\in\mathbb{R}.$$

Thus, $F_{hom}(u)\doteq\int_0^1 f_{hom}(u'(x))\,dx=\sum_{k=1}^{m}\int_{a_{k-1}}^{a_k} f_{hom}(m_k)\,dx$.

We aim to build a recovery sequence $\{\bar{u}_j\}_j$ for u. We again proceed in steps.

1) By definition of f_{hom}, $\forall k\in\{1,\dots,m\}$ there exists a competitor $v_k\in W_0^{1,p}(0,1)$ s.t.

$$f_{hom}(m_k)=\int_0^1 f(x, m_k+v_k'(x))\,dx$$

<u>N.B.</u> We may assume v_k 1-periodic $\forall k\in\mathbb{N}$.

2) Fix $k\in\{1,\dots,m\}$. $\forall x\in(a_{k-1},a_k)$ we denote $\quad u_j(x)\doteq\underbrace{m_k x+q_k}_{=u(x)}+\varepsilon_j v_k\left(\frac{x}{\varepsilon_j}\right)$, $j\in\mathbb{N}$.

Note that $v_k\left(\frac{x}{\varepsilon_j}\right)$ is fast-oscillating, but if $\varepsilon_j\searrow 0$ then $\varepsilon_j v_k\left(\frac{x}{\varepsilon_j}\right)\to 0$.
Therefore, $u_j\longrightarrow u$ in $L^p(a_{k-1},a_k)$ $\forall k\in\{1,\dots,m\}$.
Define $g(t)\doteq f(t, m_k+v_k'(t))$ $\forall t\in\mathbb{R}$. Note that g is 1-periodic by (ii) and, by a previous lemma, the sequence

$$g_j(x)\doteq g\left(\frac{x}{\varepsilon_j}\right)=f\left(\frac{x}{\varepsilon_j}, m_k+v_k'\left(\frac{x}{\varepsilon_j}\right)\right)$$

weakly converges in $L^p(0,1)$ to its average

$$\int_0^1 g(x)\,dx=\int_0^1 f(x, m_k+v_k'(x))\,dx=f_{hom}(m_k).$$

Therefore,

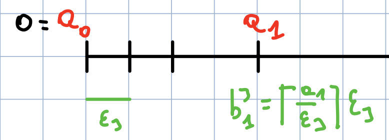
$$\int_{a_{k-1}}^{a_k} f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right)dx=\int_{a_{k-1}}^{a_k} f\left(\frac{x}{\varepsilon_j}, m_k+v_k'\left(\frac{x}{\varepsilon_j}\right)\right)dx$$

$$=\int_{a_{k-1}}^{a_k} g_j(x)\,dx\longrightarrow\int_{a_{k-1}}^{a_k} f_{hom}(m_k)\,dx \qquad \forall k\in\{1,\dots,m\}.$$

<u>Remark</u>: By (1) and (2), the sequence $\{u_j\}_j$ is a recovery sequence for u only inside the intervals of the partition in which u is affine.

If we glue all the pieces together it may be a lack of continuity

3) For any $j\in\mathbb{N}$ and $k\in\{1,\dots,m\}$ we now define $b_k^j \doteq \varepsilon_j\left\lceil \frac{a_k}{\varepsilon_j}\right\rceil$. In such way



and we denote ($\forall j\in\mathbb{N}$)

$$\overline{u}_j(x) \doteq \begin{cases} m_k x + q_k + \varepsilon_j v_k\left(\frac{x}{\varepsilon_j}\right), & \text{if } x\in\left(b_{k-1}^j+\varepsilon_j,\, b_k^j\right) \text{ for any } k\in\{1,..,m\} \\ u(x), & \text{if } x\in(0,1)\setminus \bigcup_{k=1}^{m}\left(b_{k-1}^j+\varepsilon_j,\, b_k^j\right) \doteq E_j \end{cases}$$

Then, by definition, $\{\overline{u}_j\}_j \subseteq W^{1,p}(0,1)$ and $\overline{u}_j \longrightarrow u$ strongly in $L^p(0,1)$.

Moreover,

$$F_{\varepsilon_j}(\overline{u}_j) \doteq \int_0^1 f\left(\frac{x}{\varepsilon_j}, \overline{u}_j'(x)\right)dx = \sum_{k=1}^{m}\int_{b_{k-1}^j+\varepsilon_j}^{b_k^j} f\left(\frac{x}{\varepsilon_j}, m_k + v_k'\left(\frac{x}{\varepsilon_j}\right)\right)dx + \int_{E_j} f\left(\frac{x}{\varepsilon_j}, u'(x)\right)dx$$

$\left(b_{k-1}^j+\varepsilon_j,\, b_k^j\right) \subseteq \left(a_{k-1}, a_k\right)$ and (iv)

$$\le \sum_{k=1}^{m}\int_{a_{k-1}}^{a_k} f\left(\frac{x}{\varepsilon_j}, m_k + v_k'\left(\frac{x}{\varepsilon_j}\right)\right)dx + c\int_{E_j}\left(1+\underbrace{|u'(x)|^p}_{<\infty}\right)dx \longrightarrow 0$$

since $|E_j| \to 0$ as $j \to +\infty$

$$\xrightarrow{2)} \sum_{k=1}^{m}\int_{a_{k-1}}^{a_k} f_{hom}(u'(x))\,dx = \int_0^1 f_{hom}(u'(x))\,dx = F_{hom}(u).$$

$\boxed{\text{STEP 2}}$: Let $u\in W^{1,p}(0,1)$ be any possible function.

Density argument

By the Meyers-Serrin theorem, there exists a sequence $\{u_k\}_k \subseteq W^{1,p}(0,1)$ s.t.

a) u_k is piecewise affine $\forall k\in\mathbb{N}$

b) $u_k \longrightarrow u$ strongly in $W^{1,p}(0,1)$

c) (by Step 1) $\left(\Gamma\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}\right)(u_k) \le F_{hom}(u_k)$ $\forall k\in\mathbb{N}$.

Then, by the lower semicontinuity of F_{ε_j}

$$\left(\Gamma\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}\right)(u) \overset{l.sc.}{\le} \liminf_{k\to+\infty}\left(\Gamma\text{-}\limsup_{j\to+\infty} F_{\varepsilon_j}\right)(u_k)$$

$$\overset{(c)}{\le} \liminf_{k\to+\infty} F_{hom}(u_k) = \liminf_{k\to+\infty}\int_0^1 f_{hom}(u_k'(x))\,dx$$

$f_{hom}\in C(\mathbb{R})$ + Dominated conv. th

$$= \int_0^1 f_{hom}(u'(x))\,dx = F_{hom}(u).$$

2) **Γ-liminf inequality**: $\forall u \in L^p(0,1)$ $\forall u_j \to u$ (strongly) in $L^p(0,1)$ it holds

Localization + discretization

$$F_{hom}(u) \le \liminf_{j \nearrow +\infty} F_{\varepsilon_j}(u_j) \text{ or, equivalently,}$$

$$\boxed{F_{hom}(u) \le \left(\Gamma\text{-}\liminf_{j \nearrow +\infty} F_{\varepsilon_j}\right)(u) \quad \forall u \in L^p(0,1).}$$

(In this case we will use the first statement).

Fix $u \in L^p(0,1)$ and $\{u_j\}_j \subseteq L^p(0,1)$ s.t. $u_j \to u$ strongly in $L^p(0,1)$.

We now use a **localization argument**. First, remind that

$$f_{hom}(\xi) = \lim_{T \nearrow +\infty} \min\left\{ \frac{1}{T}\int_0^T f(x, \xi + u'(x))\,dx : u \in W_0^{1,p}(0,T)\right\} \quad \text{(asymptotic hom. formula)}$$

· **Localization**: Fix $x_0 \in \mathbb{R}$, $T \in \mathbb{R}^+$ and denote

$$g_T^{x_0}(\xi) \doteq \min\left\{ \frac{1}{T}\int_{x_0}^{x_0+T} f(x, \xi + u'(x))\,dx : u \in W_0^{1,p}(x_0, x_0+T)\right\}.$$

Note that $g_T^{x_0}(\xi) \xrightarrow{T \nearrow +\infty} f_{hom}(\xi)$ uniformly (w.r.t. x_0) (*).

· Without loss of generality, assume that

$$\liminf_{j \nearrow +\infty} F_{\varepsilon_j}(u_j) < +\infty \quad \text{(otherwise the result is trivial)}$$

that is, there exists $C \in \mathbb{R}^+$ s.t. up to subsequences $F_{\varepsilon_j}(u_j) \le C$.

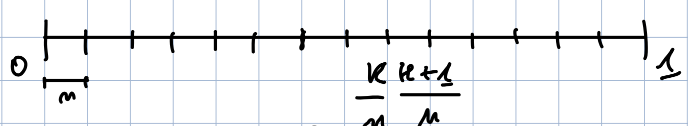
Then, by the definition of F_{ε_j}, $u_j \in W^{1,p}(0,1)$ and by the growth condition (iv)

$$C \ge F_{\varepsilon_j}(u_j) \overset{u_j \in W^{1,p}}{=} \int_0^1 f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right)dx \overset{(iv)}{\ge} \int_0^1 |u_j'(x)|^p\,dx = \|u_j'\|^p_{L^p(0,1)}.$$

Therefore, not only $u_j \to u$ strongly in $L^p(0,1)$, but also

$$u_j \rightharpoonup u \text{ weakly in } W^{1,p}(0,1) \quad \text{(up to a further subsequence).}$$

· We now need a **discretization argument**: we divide the interval $[0,1]$ in



for a fixed $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$.

Without loss of generality, by the previous lemma applied in the interval $\left(\frac{k}{m}, \frac{k+1}{m}\right)$, we assume that $u_j\left(\frac{k}{m}\right) = u\left(\frac{k}{m}\right)$ $\forall j \in \mathbb{N}$ and $\forall k \in \{0, \dots, m-1\}$.

(in the endpoints of the interval)

Then,

$$(+\infty >) \liminf_{j \nearrow +\infty} F_{\varepsilon_j}(u_j) \geq \sum_{k=0}^{m-1} \liminf_{j \nearrow +\infty} \int_{\frac{k}{m}}^{\frac{k+1}{m}} f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right) dx \quad (**).$$

Denote $z_k^m := \frac{1}{m}\left(u\left(\frac{k+1}{m}\right) - u\left(\frac{k}{m}\right)\right) = \frac{1}{m}\left(u_j\left(\frac{k+1}{m}\right) - u_j\left(\frac{k}{m}\right)\right)$ and consider the lines

$$w_j(x) := u_j(x) - \left[u_j\left(\frac{k}{m}\right) + z_k^m \cdot \left(x - \frac{k}{m}\right)\right] \quad \text{for any } x \in \left(\frac{k}{m}, \frac{k+1}{m}\right).$$

By construction, $w_j \in W_0^{1,p}\left(\frac{k}{m}, \frac{k+1}{m}\right)$, and

$$\int_{\frac{k}{m}}^{\frac{k+1}{m}} f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right) dx = \int_{\frac{k}{m}}^{\frac{k+1}{m}} f\left(\frac{x}{\varepsilon_j}, w_j'(x) + z_k^m\right) dx$$

Change of variable: $\frac{x}{\varepsilon_j} =: s$

$\tilde{w}_j(s) := \frac{1}{\varepsilon_j} w_j(\varepsilon_j s)$

$\tilde{w}_j \in W_0^{1,p}\left(\frac{k}{\varepsilon_j m}, \frac{k+1}{\varepsilon_j m}\right)$

and $\tilde{w}_j'(s) = w_j'(\varepsilon_j s) = w_j'(x)$

$$= \varepsilon_j \cdot \int_{\frac{k}{m\varepsilon_j}}^{\frac{k+1}{m\varepsilon_j}} f\left(s, \tilde{w}_j'(s) + z_k^m\right) ds$$

$$\geq \min\left\{\varepsilon_j \cdot \int_{\frac{k}{m\varepsilon_j}}^{\frac{k+1}{m\varepsilon_j}} f\left(s, w'(s) + z_k^m\right) ds : w \in W_0^{1,p}\left(\frac{k}{\varepsilon_j m}, \frac{k+1}{\varepsilon_j m}\right)\right\}$$

$$= \frac{1}{m} \min\left\{m\,\varepsilon_j \int_{\frac{k}{m\varepsilon_j}}^{\frac{k+1}{m\varepsilon_j}} f\left(s, w'(s) + z_k^m\right) ds : w \in W_0^{1,p}\right\}$$

Let $T := \frac{1}{m\varepsilon_j}$ and $x_0 := \frac{k}{m\varepsilon_j}$

$$= \frac{1}{m}\, g_T^{x_0}(z_k^m)$$

and so, by $(*)$ and $(**)$,

$$\liminf_{j \nearrow +\infty} F_{\varepsilon_j}(u_j) \overset{(**)}{\geq} \sum_{k=0}^{m-1} \liminf_{j \nearrow +\infty} \int_{\frac{k}{m}}^{\frac{k+1}{m}} f\left(\frac{x}{\varepsilon_j}, u_j'(x)\right) dx$$

$$\geq \sum_{k=0}^{m-1} \liminf_{j \nearrow +\infty} \frac{1}{m}\, g_{\frac{1}{\varepsilon_j m}}^{\frac{k}{\varepsilon_j m}}(z_k^m)$$

$$\overset{(*)}{=} \frac{1}{m} \sum_{k=0}^{m-1} f_{hom}\Big(\underbrace{z_k^m}_{\text{slope of the line } w_j(x)}\Big) = \int_0^1 f_{hom}\left(\tilde{u}_m'(x)\right) dx,$$

where $\tilde{u}_n'(x) = z_k^n$ is the interpolation line of u in the points $\frac{k}{n}$
$\left(x \in \left(\frac{k}{n}, \frac{k+1}{n}\right)\right)$

and $\tilde{u}_n$ is also piecewise affine.

Then, $\liminf_{j \nearrow +\infty} F_{\varepsilon_j}(u_j) \geq F_{hom}(\tilde{u}_n) \xrightarrow{n \nearrow +\infty} F_{hom}(u)$ (being $\tilde{u}_n$ strongly convergent to u in $W^{1,p}(0,1)$). $\square$