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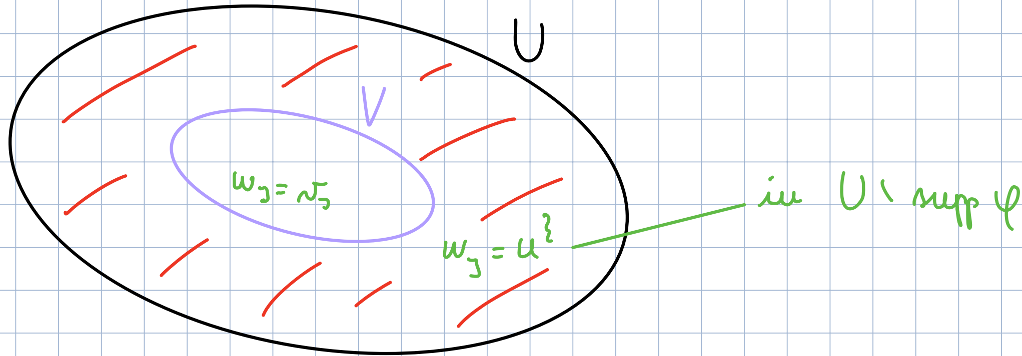
STEP 2 We conclude with a localization argument.

Fix $U \subseteq \Omega$ open and let $V \subset\subset U$ be open ($\overline{V} \subseteq U$). Moreover, let φ be a cut-off function (i.e. $\varphi \in C_c^\infty(U)$, $\varphi = 1$ in V, $0 \le \varphi \le 1$ in $U \setminus V$) and let $w_j = \varphi v_j + (1-\varphi) u^\xi$.



By Step 1, $w_j \xrightarrow{*} u^\xi$ in $W^{1,\infty}$-weak $*$ and, by hypotheses,

∘ $$F(u^\xi) \le \liminf_{j \to +\infty} F(w_j).$$

∘∘ · Note that $F(u^\xi) = \int_\Omega f(x,\xi)\,dx = \int_{\Omega \setminus U} f(x,\xi)\,dx + \int_U f(x,\xi)\,dx$

and

∘∘∘ $$F(w_j) = \int_\Omega f(x, \nabla w_j(x))\,dx = \int_V f(x, \nabla v_j(x))\,dx + \int_{U\setminus V} f(x, \varphi \nabla v_j + (1-\varphi)\xi + \nabla\varphi \cdot (v_j - u^\xi))\,dx + \int_{\Omega\setminus U} f(x,\xi)\,dx$$

$$\Rightarrow \int_U f(x,\xi)\,dx \le \liminf_{j\to+\infty} \left[\int_V f(x,\nabla v_j(x))\,dx + \int_{U\setminus V} f(x, \varphi\nabla v_j + (1-\varphi)\xi + \nabla\varphi\cdot(v_j - u^\xi))\,dx \right]$$

· By Step 1, $\int_V f(x, \nabla v_j(x))\,dx \longrightarrow t\int_V f(x,\xi_1)\,dx + (1-t)\int_V f(x,\xi_2)\,dx$.

Consider now the term $\int_{U\setminus V} f(x, \varphi\nabla v_j + (1-\varphi)\xi + \nabla\varphi\cdot(v_j - u^\xi))\,dx$.

By construction, $\varphi \in [0,1]$, $\xi \in \mathbb{R}^m$ is a fixed vector and $\nabla v_j \in \{\xi_1, \xi_2\}$. Then, $\exists\, M \in \mathbb{R}^+$ s.t.

- $\|\varphi\nabla v_j + (1-\varphi)\xi\|_{L^\infty} \le M$ and
- $\|\nabla\varphi\cdot(v_j - u^\xi)\|_{L^\infty} \overset{C\text{-}S}{\le} \|\nabla v_j\|_{L^\infty} \|v_j - u^\xi\|_{L^\infty} \le 1$ for j large enough, being $v_j \xrightarrow{*} u^\xi$.

it depends on U and V

$\Rightarrow \|\varphi\nabla v_j + (1-\varphi)\xi + \nabla\varphi\cdot(v_j - u^\xi)\|_{L^\infty} \le M+1$ and so there exists $g_{M+1} \in L^1(\Omega)$ s.t.

$$f(x, \varphi\nabla v_j + (1-\varphi)\xi + \nabla\varphi\cdot(v_j - u^\xi)) \le g_{M+1}(x) \text{ a.e. } x \in \Omega \text{ for } j \text{ large enough.}$$

Then, $\int_U f(x,\eta)\,dx \le t\int_V f(x,\xi_1)\,dx + (1-t)\int_V f(x,\xi_2)\,dx + \int_{U\setminus V} g_{M+1}(x)\,dx.$

Since g_{M+1} is independent of the choice of U and V, once we push V to approach U
we consider the sup $V\subset\subset U$

then $\int_{U\setminus V} g_{m+1}(x)\,dx \longrightarrow 0$, that is

$$\int_U f(x,\eta)\,dx \le t\int_U f(x,\xi_1)\,dx + (1-t)\int_U f(x,\xi_2)\,dx \quad \forall\, U\subseteq\Omega \text{ open.}$$

· Since it holds for any open set U, then

$$\boxed{\int_U f(x,\eta)\,dx \le t\int_U f(x,\xi_1)\,dx + (1-t)\int_U f(x,\xi_2)\,dx}$$

$\forall\, \xi_1,\xi_2\in\mathbb{R}^n$, $\forall\, t\in(0,1)$, $\forall\, U\subseteq\Omega$ open, and so $\exists\, N = N(\xi_1,\xi_2,t)\subseteq\Omega$ s.t. $|N| = 0$ (Lebesgue measure) and s.t.

$$(**) \qquad f(x,\xi) \le t\, f(x,\xi_1) + (1-t)\, f(x,\xi_2) \quad \forall\, x\in\Omega\setminus N.$$

<u>Remark</u>: N depends on ξ_1,ξ_2 and t and so we cannot simple consider the union of all N (in $\mathbb{R}^n$ and $\mathbb{R}$), because it may have positive Lebesgue measure.
We should then work with a countable union of sets.

Denote $\tilde{N} \doteq \bigcup_{\substack{\xi_1,\xi_2\in\mathbb{Q}^n \\ t\in(0,1)\cap\mathbb{Q}}} N(\xi_1,\xi_2,t)$. Then, $|\tilde{N}| = 0$ and $(**)$ holds true $\forall\, x\in\Omega\setminus\tilde{N}$.

The conclusion then follows being, by hypotheses, $f(x,\cdot)$ continuous a.e. $x\in\Omega$.

$\square$

1.1) Hogenization in dimension $n=1$

In dimension $n=1$, our class of energies depending on $\varepsilon\in\mathbb{R}^+$ becomes

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, u'(x)\right) dx,$$

where Ω is the interval $(0,1)$ and f is periodic w.r.t. the first variable.

Our aim is to study Γ-convergence for the sequence of functionals $\{F_\varepsilon\}_\varepsilon$ as $\varepsilon \to 0$, and w.r.t. a proper topology. The problem can be formulated as:

<u>Problem</u>: Fix $1<p<+\infty$ and $f : \mathbb{R}\times\mathbb{R} \longrightarrow [0,+\infty]$ satisfying

i) f is a Carathéodory function;

ii) $\forall\, \xi\in\mathbb{R}$ the map $f(\cdot,\xi)$ is 1-periodic;

iii) for a.e. $x\in\mathbb{R}$ the map $f(x,\cdot)$ is convex; F_ε l.sc. in $W^{1,p}$

iv) there exists $c\in\mathbb{R}^+$ s.t. $|\xi|^p \leq f(x,\xi) \leq c\,(|\xi|^p+1)$, $(p>1)$

F_ε coercive $\qquad F_\varepsilon<+\infty$ in $W^{1,p}$

and define the sequence $F_\varepsilon : L^p(0,1) \longrightarrow [0,+\infty]$ $\quad(\varepsilon\in\mathbb{R}^+)$

$$u \longmapsto \begin{cases} \int_0^1 f\left(\frac{x}{\varepsilon}, u'(x)\right)dx, & \text{if } u\in W^{1,p}(0,1) \\ +\infty & \text{, otherwise} \end{cases}$$

Q: There exists $f_{hom} : \mathbb{R} \longrightarrow [0,+\infty)$ convex s.t. $\{F_\varepsilon\}_\varepsilon$ Γ-converges to

$F_{hom} : L^p(0,1) \longrightarrow [0,+\infty]$

$$u \longmapsto \begin{cases} \int_0^1 f_{hom}(u'(x))\,dx, & \text{if } u\in W^{1,p}(0,1) \\ +\infty & \text{, otherwise} \end{cases}$$

in the strong topology of $L^p(0,1)$?

<u>Remark</u>: 1) f_{hom} convex implies F_{hom} convex. It is crucial to have convexity for the limit being $\{F_\varepsilon\}_\varepsilon$ a sequence of convex functionals and since Γ-convergence preserves convexity.

2) f_{hom} convex implies F_{hom} l.s.c.. We have proved that the convexity of the integrand holds if and only if the integral functional is l.sc.

3) Q: What about equi-coercivity?

· Fix $\varphi\in W^{1,p}(0,1)$ and let $\{u_\varepsilon\}_\varepsilon \subseteq L^p(0,1)$ satisfy $\sup_\varepsilon \tilde F_\varepsilon(u_\varepsilon)<+\infty$, where

$$\tilde F_\varepsilon(u) = \begin{cases} \int_0^1 f\left(\frac{x}{\varepsilon}, u'(x)\right)dx, & \text{if } u\in W^{1,p}_\varphi(0,1) \doteq \{v\in W^{1,p}(0,1) \text{ s.t. } v-\varphi\in W^{1,p}_0(0,1)\} \\ +\infty & \text{, if } u\in L^p(0,1)\setminus W^{1,p}_\varphi(0,1). \end{cases}$$

Then, necessarily $u_\varepsilon \in W^{1,p}_\varphi(0,1)$ $\forall \varepsilon \in \mathbb{R}^+$ and, by (iv),

$$+\infty > \tilde{F}_\varepsilon(u_\varepsilon) = \int_0^1 f_\varepsilon\left(\frac{x}{\varepsilon}, u_\varepsilon'(x)\right) dx \overset{(iv)}{\geq} \int_0^1 |u_\varepsilon'(x)|^p dx \qquad \forall \varepsilon \in \mathbb{R}^+$$

$\Rightarrow \sup_\varepsilon \|u_\varepsilon'\|_{L^p} < +\infty$. (*)

Note that $\|u_\varepsilon\|_{L^p} \overset{tr.}{\leq} \|u_\varepsilon - \varphi\|_{L^p} + \|\varphi\|_{L^p}$, by the triangular inequality.

Moreover, since $u_\varepsilon \in W^{1,p}_\varphi(0,1)$, i.e. $u_\varepsilon - \varphi \in W^{1,p}_0(0,1)$, then by the Poincaré ineq. there exist $c \in \mathbb{R}^+$ s.t. $\|u_\varepsilon - \varphi\|_{L^p} \leq c\|u_\varepsilon' - \varphi'\|_{L^p}$ and so

$$\|u_\varepsilon\|_{L^p} \leq c\|u_\varepsilon' - \varphi'\|_{L^p} + \|\varphi\|_{L^p} \leq C_1 \quad \text{constant.} \quad (**)$$

By (*) and (**), we finally get $\sup_\varepsilon \|u_\varepsilon\|_{W^{1,p}} < +\infty$ and so the sequence $\{u_\varepsilon\}_\varepsilon$ is bounded in $W^{1,p}(0,1)$, reflexive being $p>1$, that is $\exists \{u_{\varepsilon_j}\}_j \subseteq \{u_\varepsilon\}_\varepsilon$ and $u \in W^{1,p}(0,1)$ s.t.

a) $u_{j_\varepsilon} \rightharpoonup u$ in $W^{1,p}(0,1)$ - weakly

b) $u_{j_\varepsilon} \to u$ in $L^p(0,1)$ - strongly (by the compact embedding)

4) Q: What can we say about f_{hom} (a part from the convexity)?

· Let us first recall a generalization of the convexity condition for Banach spaces.

Prop: (Jensen's inequality) Let $(X, \|\cdot\|_X)$ be a Banach space and let $F: X \to [0,+\infty]$ be convex and lower semicontinuous. Moreover, let $(E, \mathcal{E}, \mu)$ be a measure space with $\mu \geq 0$ and $\mu(E) = 1$. Then,

$$\boxed{F\left(\int_E u(s)\, d\mu(s)\right) \leq \int_E F(u(s))\, d\mu(s)} \qquad \forall u \in L^1_\mu(E; X).$$

Remark: if $\mu(E) \neq 1 \Rightarrow F\left(\frac{1}{\mu(E)} \int_E u\, d\mu\right) \leq \frac{1}{\mu(E)} \int_E F(u)\, d\mu \quad \forall u \in L^1_\mu(E;X)$.

· Assume for a moment that $\{F_\varepsilon\}_\varepsilon$ Γ-converges to F_{hom}, defined above.

Fix $\xi \in \mathbb{R}^n$. Then, by the Jensen's inequality for any $u \in W^{1,p}_0(0,1)$

$$\int_0^1 f_{hom}(\xi + u'(x))\, dx \overset{J}{\geq} f_{hom}\left(\int_0^1 (\xi + u'(x))\, dx\right)$$

$$= f_{hom}(\xi + (u(1) - \cancel{u(0)})) = f_{hom}(\xi)$$

$u \in W^{1,p}_0$

$$\Rightarrow f_{hom}(\xi) = \min\left\{ \int_0^1 f_{hom}(\xi + u'(x))\,dx \,:\, u \in W_0^{1,p}(0,1) \right\}$$

and, by the fundamental theorem of Γ-convergence

$$= \lim_{\varepsilon \searrow 0} \min\left\{ \int_0^1 f\left(\frac{x}{\varepsilon}, \xi + u'(x)\right)dx : u \in W_0^{1,p}(0,1) \right\}$$

$T \doteq \frac{1}{\varepsilon}$, $s \doteq \frac{x}{\varepsilon}$, $v \doteq \frac{u(\varepsilon s)}{\varepsilon}$

$$= \lim_{T \nearrow +\infty} \min\left\{ \frac{1}{T}\int_0^T f(s, \xi + v'(s))\,ds : v \in W_0^{1,p}(0,T) \right\}$$

$$\Rightarrow \boxed{f_{hom}(\xi) = \lim_{T \nearrow +\infty} \min\left\{ \frac{1}{T}\int_0^T f(s, \xi + v'(s))\,ds : v \in W_0^{1,p}(0,T) \right\}}$$

Def: The previous formula is called **asymptotic homogenization formula**.

Another interesting formulation of the problem happens when u is 1-periodic:

$$f_{hom}(\xi) = \min\left\{ \int_0^1 f_{hom}(\xi + u'(x))\,dx : u \in W^{1,p}(0,1) \text{ and } u(0) = u(1) \right\}$$

it is enough to arrive at this point

$\varepsilon_k = \frac{1}{k}$, $k \in \mathbb{N}$, $v \doteq \frac{u(\varepsilon_k s)}{\varepsilon}$

$$= \lim_{k \nearrow +\infty} \min\left\{ \frac{1}{k}\int_0^k f(s, \xi + v'(s))\,ds : v \in W^{1,p}(0,k) \text{ and } v(0) = v(k) \right\}.$$

CLAIM: Denote $M_k(\xi)$ the previous minimization problem. Then, $M_k(\xi) = M_1(\xi)$ $\forall k \in \mathbb{N}$

. Once it holds true, then

$$\boxed{f_{hom}(\xi) = \min\left\{ \int_0^1 f(s, \xi + u'(s))\,ds : u \in W^{1,p}(0,1) \text{ and } u(0) = u(1) \right\}.}$$

Def: The previous formulation is called **cell problem formula**.

Proof (claim):

(Step 1 $M_k \leq M_1$)

Let $u \in W^{1,p}(0,1)$ satisfy $u(0) = u(1)$. We extend u in all $(0,k)$ by periodicity

$$\Rightarrow \frac{1}{k}\int_0^k f(s, \xi + u'(s))\,ds \underset{f,u \text{ 1-periodic}}{=} \int_0^1 f(s, \xi + u'(s))\,ds \Rightarrow M_k(\xi) \leq M_1(\xi).$$

(Step$_2$ $M_k \geq M_1$) Let $u \in W^{1,p}(0,k)$ s.t. $u(0) = u(k)$, with $k \in \mathbb{N}$ fixed.

To get a 1-periodic function, we study the convex combination of the translations of u.

Denote $v(x) \doteq \sum_{i=0}^{k-1} \frac{1}{k} u(x+i)$. Then, $v \in W^{1,p}(0,1)$ and it is 1-periodic ($v(0) = v(1)$)

$$\Rightarrow M_1(\xi) \overset{\min}{\leq} \int_0^1 f(s, \xi + v'(s))\,ds \overset{f,\, v\ 1\text{-p.}}{=} \frac{1}{k}\int_0^k f(s, \xi + v'(s))\,ds$$

$$\overset{\text{def}}{=} \frac{1}{k}\int_0^k f\Big(s, \xi + \sum_{i=0}^{k-1} \frac{1}{k} u'(s+i)\Big)\,ds \overset{f \text{ convex}}{\leq} \frac{1}{k}\sum_{i=0}^{k-1} \frac{1}{k}\int_0^k f(s, \xi + u'(s+i))\,ds$$

$$\overset{f\ 1\text{-p.}}{=} \frac{1}{k}\sum_{i=0}^{k-1} \frac{1}{k}\int_0^k f(s+i, \xi + u'(s+i))\,ds \overset{u\ k\text{-p.}}{=} \frac{1}{k}\sum_{i=0}^{k-1} \frac{1}{k}\int_0^k f(s, \xi + u'(s))\,ds$$

$$= \frac{1}{k}\int_0^k f(s, \xi + u'(s))\,ds$$

and, passing to the minimum, we finally get $M_1(\xi) \leq M_k(\xi)$.

$\square$