19.11.2024

Consider now the minimization problem

(P) $$\boxed{\min\left\{F(u) \ : \ u\in W_0^{1,p}(\Omega)\right\}.}$$

We showed that if $p>1$ (reflexivity of the space) and if we consider the space with zero boundary datum $W_0^{1,p}(\Omega)$, instead of the whole space $W^{1,p}(\Omega)$, then the growth condition on f ensures the coercivity of F.
Let us then recollect all the previous considerations in the following result.

Th: Let $\Omega\subseteq\mathbb{R}^n$ be open and bounded, let $1<p<+\infty$ and consider
$f:\Omega\times\mathbb{R}^n\longrightarrow[0,+\infty]$ satisfying

i) f is Borel-measurable;

ii) for a.e. $x\in\Omega$ the map $\begin{aligned}\mathbb{R}^n&\longrightarrow[0,+\infty]\\ \xi&\longmapsto f(x,\xi)\end{aligned}$ is lower semicontinuous;

iii) for a.e. $x\in\Omega$ the map $\begin{aligned}\mathbb{R}^n&\longrightarrow[0,+\infty]\\ \xi&\longmapsto f(x,\xi)\end{aligned}$ is convex;

iv) there exists $c_1\in\mathbb{R}^+$ s.t. $f(x,\xi)\geq c_1|\xi|^p$ a.e. $x\in\Omega$ $\forall\xi\in\mathbb{R}^n$.

Then, problem (P) admits solutions.

Proof: Let $\{u_j\}_j\subseteq W_0^{1,p}(\Omega)$ be a minimizing sequence for F, i.e.

$$\lim_{j\to+\infty}F(u_j)=\inf_{W_0^{1,p}}F\ (<+\infty).$$

Then, $\sup_{j\in\mathbb{N}}F(u_j)<+\infty$ and, by (iv)

$$+\infty>F(u_j)\doteq\int_\Omega f(x,\nabla u_j)\,dx\geq c_1\|\nabla u_j\|^p_{L^p(\Omega)}\qquad\forall j\in\mathbb{N},$$

that is, $\sup_{j\in\mathbb{N}}\|\nabla u_j\|_{L^p(\Omega)}<+\infty$.

Therefore, as a consequence of the Poincaré inequality,

$$\sup_{j\in\mathbb{N}} \|u_j\|_{W^{1,p}(\Omega)} < +\infty \Rightarrow \{u_j\}_j \text{ is a bounded sequence in } W_0^{1,p}(\Omega)$$

and, by reflexivity $(p>1)$ $\exists\ u \in W^{1,p}(\Omega)$ and $\exists \{u_{j_k}\}_k \subseteq \{u_j\}_j$ s.t.

$$u_{j_k} \longrightarrow u \quad \text{weakly in } W^{1,p}(\Omega).$$

limit of conv. cond. is convex

Note that, in principle, since $W_0^{1,p}(\Omega)$ inherits its topology from $W^{1,p}(\Omega)$, then

$$u \in W^{1,p}(\Omega) \quad (u \text{ is the candidate of minimizer}).$$

Since $\overline{W_0^{1,p}(\Omega)}^{\|\cdot\|_{W^{1,p}}} = W_0^{1,p}(\Omega)$ (its strongly closed) and $W_0^{1,p}(\Omega)$ is convex, then $W_0^{1,p}(\Omega)$ is also closed in the weak topology of $W^{1,p}(\Omega)$ and so $u \in W_0^{1,p}(\Omega)$.

By the previous result, $F(u) \overset{F\ \text{l.s.c.}}{\leq} \liminf_{k\to+\infty} F(u_{j_k}) = \inf_{W_0^{1,p}} F.$ $\square$

Ex: Prove the following result

Th: Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, such that $\partial\Omega$ is Lipschitz continuous, let $1<p<+\infty$ and consider $F(u) \doteq \int_\Omega f(x, \nabla u(x))\,dx$, where f satisfy the assumptions of the previous theorem, and let $g: \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfy

i) g is Borel-measurable;

ii) for a.e. $x\in\Omega$ the map $\begin{array}{rcl}\mathbb{R} & \longrightarrow & \mathbb{R} \\ s & \longmapsto & g(x,s)\end{array}$ is lower semicontinuous;

iii) there exists $c_2 \in \mathbb{R}^+$ s.t. $g(x,s) \geq c|s|^p$ a.e. $x\in\Omega$ $\forall s\in\mathbb{R}$.

Then, the following problem admits solution

$$\min\left\{F(u)+G(u) : u \in W^{1,p}(\Omega)\right\}$$

where $G(u) \doteq \int_\Omega g(x, u(x))\,dx$.

(Hint) In $W^{1,p}(\Omega)$ it does not hold the Poincaré inequality. Use the Rellich theorem.

· In what follows, we want to show that the convexity condition on $f(x,\cdot)$ a.e. $x\in\Omega$ is not only a sufficient condition to get the lower semicontinuity of F, but it is also necessary. We first need the following result.

Lemma: Let $g \in L^\infty(\mathbb{R})$ be 1-periodic (i.e. $g(x+1) = g(x)$ $\forall x \in \mathbb{R}$) and let $\{g_j\}_j \subseteq L^\infty(\mathbb{R})$ satisfy $g_j(x) = g(jx)$ $\forall j \in \mathbb{N}$.
Then, $g_j \xrightarrow{*} m \doteq \int_0^1 g(x)\,dx$ (average) in the weak-* topology of L^∞.

Remark: Note if the period is different from 1, then the result still holds true by replacing m with $\tilde{m} = \dfrac{\int_{\text{per}} g\,dx}{(\text{length of the period})}$.

Proof: Without loss of generality we assume $m = 0$ (it is not restrictive. Otherwise, we consider $\tilde{g}(x) = g(x) - m$ $\forall x \in \mathbb{R}$).

GOAL: $\int_{\mathbb{R}} g_j(x)\varphi(x)\,dx \longrightarrow \int_{\mathbb{R}} m(x)\varphi(x)\,dx = 0$ $\forall \varphi \in L^1(\mathbb{R})$.

N.B.: $C_c^1(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ $\forall 1 \leq p < +\infty$. We can then replace $\varphi \in L^1(\mathbb{R})$ with a test function $\varphi \in C_c^1(\mathbb{R})$.

Let $G(x) \doteq \int_0^x g(t)\,dt$ $\forall x \in \mathbb{R}$.

N.B.: $g \in L^\infty(\mathbb{R}) \Rightarrow g$ is measurable in $\mathbb{R}$ and bounded
$\Rightarrow G$ is well-defined, G is differentiable a.e. in $\mathbb{R}$
$\Rightarrow G'(x) = g(x)$.

g is periodic $+ \int_0^1 g(x)\,dx = 0$ $\Rightarrow G$ is 1-periodic $\Rightarrow G$ is bounded.

$m=0$

Ex: Show that G is 1-periodic.

Let $H_j(x) \doteq G(jx)$. Then, H_j is differentiable a.e. in $\mathbb{R}$ $\forall j \in \mathbb{N}$ and

$$H_j'(x) \doteq G'(jx) = j\,g(jx) \overset{\text{Hyp}}{=} j\,g_j(x) \quad \text{a.e. } x \in \mathbb{R} \ \forall j \in \mathbb{N}.$$

And so

$$\left|\int_{\mathbb{R}} g_j(x)\varphi(x)\,dx\right| = \frac{1}{j}\left|\int_{\mathbb{R}} H_j'(x)\varphi(x)\,dx\right| \overset{\varphi \in C_c^1}{=} \left|-\frac{1}{j}\int_{\mathbb{R}} \overbrace{H_j(x)}^{\text{bounded}}\varphi'(x)\,dx\right| \leq \frac{C}{j} \longrightarrow 0.$$

$\square$

- We are now in the position to prove the next result.

Th: (Necessity of convexity)

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and let $f : \Omega \times \mathbb{R}^n \longrightarrow [0,+\infty)$ satisfy:

i) f is a Carathéodory function (i.e. $f(\cdot,\xi)$ is measurable $\forall \xi \in \mathbb{R}^n$ and $f(x,\cdot)$ is continuous a.e. $x \in \Omega$);

technical request

ii) $\forall R \in \mathbb{R}^+ \; \exists g_R \in L^1(\Omega)$ s.t. $0 \le f(x,\xi) \le g_R(x)$ a.e. $x \in \Omega$ $\forall |\xi| \le R$.

Moreover, consider the functional

$$F : W^{1,\infty}(\Omega) \longrightarrow \mathbb{R}, \quad u \longmapsto F(u) \doteq \int_\Omega f(x, \nabla u(x))\,dx$$

- If F is sequentially lower semicontinuous (w.r.t. the $W^{1,\infty}$- weak $*$ topology), then for a.e. $x \in \Omega$ the map $\mathbb{R}^n \longrightarrow \mathbb{R}$, $\xi \longmapsto f(x,\xi)$ is convex.

Remark: Note that the $W^{1,\infty}$- weak $*$ assumption is the weakest one in the following sense:

EX: if F is sequentially lower semicontinuous w.r.t. the $W^{1,p}$- weak topology for some $1 \le p < +\infty$, then F is sequentially lower semicontinuous w.r.t. the $W^{1,\infty}$- weak $*$ topology.

(Hint: $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}$ $\Rightarrow$ $u_j \rightharpoonup u$ in $W^{1,p}$ $\forall\, 1 \le p < +\infty$)

Proof: Fix $\xi_1, \xi_2 \in \mathbb{R}^n$ and $t \in (0,1)$, and denote $\xi \doteq t\xi_1 + (1-t)\xi_2$. We want to show that $f(x,\xi) \le t f(x,\xi_1) + (1-t) f(x,\xi_2)$ a.e. $x \in \Omega$.

- The proof is divided in several steps.

STEP1 Given the affine function $u^\xi \in C^\infty(\Omega)$, associated with ξ and defined by

$$u^\xi(x) \doteq \langle x, \xi \rangle \doteq \sum_{i=1}^n x_i \xi_i \quad \text{for any } x \in \Omega \quad (\nabla u^\xi = \xi \; *),$$

we want to build by hand a sequence $\{u_j\}_j \subseteq W^{1,\infty}(\Omega)$ s.t.

1) $u_j \overset{*}{\rightharpoonup} u^\xi$ in $W^{1,\infty}$- weak $*$

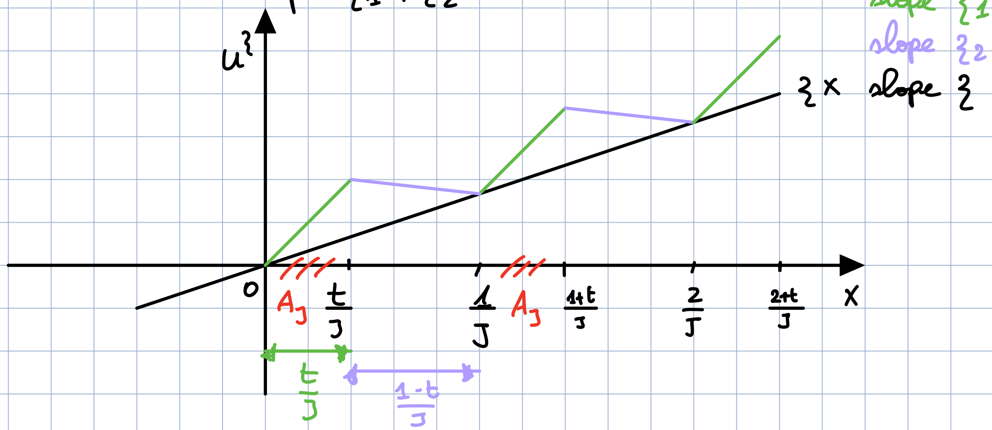
2) $F(u_j) \longrightarrow t\int_\Omega f(x,\xi_1)\,dx + (1-t)\int_\Omega f(x,\xi_2)\,dx$ as $j \to +\infty$.

In fact, by hypotheses, we get the following integral version of the convexity

$$\int_\Omega f(x,\xi)\,dx \overset{*}{=} F(u^\xi) \overset{(1)}{\le} \liminf_{j\to+\infty} F(u_j) \overset{(2)}{=} t\int_\Omega f(x,\xi_1)\,dx + (1-t)\int_\Omega f(x,\xi_2)\,dx$$

STEP 1.1 Case $n=1$. We want to approximate the line $u^\xi = \langle x, \xi \rangle = \xi x$ $(n=1)$

by means of a microstructure of period J, only depending on the slops ξ_1, ξ_2.



Remark: We actually do not need to know the analytic expression of u_J.

- Denote $A_J \doteq \{x \in \Omega : u_J'(x) = \xi_1\} = \left[0, \frac{t}{J}\right] \cup \left[\frac{1}{J}, \frac{1}{J} + \frac{t}{J}\right] \cup \dots$

  and consider the characteristic function of A_J

$$\chi_{A_J}(x) \doteq \begin{cases} 1 & , \text{ if } x \in A_J \\ 0 & , \text{ if } x \in \mathbb{R} \setminus A_J \end{cases}.$$

By periodicity, $\chi_{A_J}(x) = \chi_{A_1}(Jx) = \chi_{[0,t]}(x)$ extended by periodicity.

Then, by the previous lemma,

- $\chi_{A_J} \overset{*}{\rightharpoonup} m \doteq \int_0^1 \chi_{A_1}(x)\,dx = \int_0^t dx = t$ in L^∞-weak $*$
- $\chi_{\mathbb{R} \setminus A_J} \overset{*}{\rightharpoonup} 1-t$ in L^∞-weak $*$

and so $u_J' \doteq \xi_1 \chi_{A_J} + \xi_2 \chi_{\mathbb{R}\setminus A_J} \overset{*}{\rightharpoonup} t\xi_1 + (1-t)\xi_2 \doteq \xi = (u^\xi)'$ in L^∞-weak $*$,

which implies (by construction $u_J \overset{*}{\rightharpoonup} u^\xi$ in L^∞-weak $*$)

$$u_J \overset{*}{\rightharpoonup} u^\xi \text{ in } W^{1,\infty}\text{-weak } *.$$

Then, $F(u_J) \doteq \int_\Omega f(x, u_J'(x))\,dx = \int_{A_J} f(x,\xi_1)\,dx + \int_{\mathbb{R}\setminus A_J} f(x,\xi_2)\,dx$

$$= \int_\Omega \underbrace{f(x,\xi_1)}_{\in L^1} \underbrace{\chi_{A_J}(x)}_{\overset{*}{\rightharpoonup} t}\,dx + \int_\Omega \underbrace{f(x,\xi_2)}_{\in L^1} \underbrace{\chi_{\mathbb{R}\setminus A_J}(x)}_{\overset{*}{\rightharpoonup} 1-t}\,dx \quad \text{in } L^\infty\text{-weak}*$$

We can find $R \in \mathbb{R}^+$ s.t. $f(x,\xi) \le g_R(x) \in L^1(\Omega)$

and so $F(u_J) \longrightarrow t \int_\Omega f(x,\xi_1)\,dx + (1-t)\int_\Omega f(x,\xi_2)\,dx.$

STEP 1.2 Case $n > 1$ Let $\xi = t\,\xi_1 + (1-t)\,\xi_2$, with $\xi_1, \xi_2 \in \mathbb{R}^n$ such that $\xi_1 - \xi_2 \,//\, e_1$.

For any $x \in \mathbb{R}^n$ we use the following notation: $x = (\bar{x}, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$.

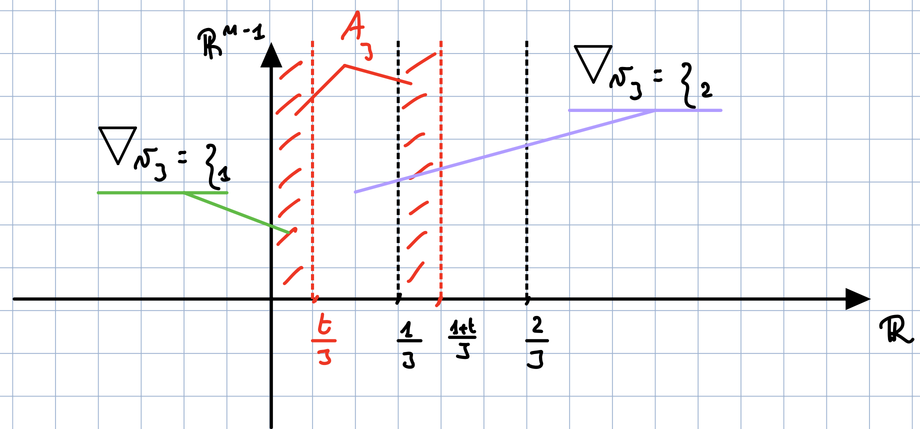
Then, $\xi = (\bar{\xi}, \xi')$ s.t. $\bar{\xi} = t\,\bar{\xi}_1 + (1-t)\,\bar{\xi}_2$ and $\xi' = t\,\xi_1' + (1-t)\,\xi_2'$.

Since we assumed $\xi_1 - \xi_2 = (\bar{\xi}_1 - \bar{\xi}_2, \xi_1' - \xi_2') \,//\, e_1 = (1, 0)$, then $\xi_1' = \xi_2'$.

We then define $v_J(x) = \underline{u_J(\bar{x})} + \langle x', \xi' \rangle_{\mathbb{R}^{n-1}}$ and, by step 1.1

from step 1.1

$$v_J \overset{*}{\rightharpoonup} \bar{x}\,\bar{\xi} + \langle x', \xi' \rangle_{\mathbb{R}^{n-1}} = \langle x, \xi \rangle = u^{\xi} \quad \text{in } W^{1,\infty}\text{-weak } *.$$



(it is the analogous of the construction did in Step 1.1, now along the first coordinate)

Like before, we denote $A_J = \{ x \in \mathbb{R}^n : \nabla v_J = \xi_1 \}$ and, by the previous step,

- $\chi_{A_J} \overset{*}{\rightharpoonup} t$ in L^∞-weak $*$
- $\chi_{\mathbb{R}^n \setminus A_J} \overset{*}{\rightharpoonup} 1-t$ in L^∞-weak $*$

and so

$$F(v_J) = \int_\Omega f(x, \nabla v_J(x))\,dx = \int_\Omega f(x, \xi_1)\,\chi_{A_J}(x)\,dx + \int_\Omega f(x, \xi_2)\,\chi_{\mathbb{R}^n \setminus A_J}(x)\,dx$$

$$\longrightarrow t \int_\Omega f(x, \xi_1)\,dx + (1-t) \int_\Omega f(x, \xi_2)\,dx.$$