3) Application to elliptic PDEs: the G-convergence problem

05.12.2024

We now apply the Γ-convergence theory to functional F_ε that correspond to the energy functionals associated with elliptic PDEs in divergence form

Ex: (dim $n=1$)

$$(P)_\varepsilon \begin{cases} A_\varepsilon v = 0 & \text{in } (0,1) \\ v(0)=0 \\ v(1)=1 \end{cases}$$

where the second order operator in divergence form A_ε is defined by

$$A_\varepsilon : W^{1,2}(0,1) \longrightarrow \left(W^{1,2}(0,1)\right)'$$

$$v \longmapsto A_\varepsilon(v) = -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right) v'(x)\right) = -\left(a\left(\frac{x}{\varepsilon}\right) v'\right)'$$

for any $\varepsilon \in \mathbb{R}^+$.

By the standard theory of PDEs (e.g. Lax-Milgram Lemma), any minimizer of F_ε, $\bar{v} \in W^{1,2}(0,1)$, is also the unique (up to constants) weak solution to $(P)_\varepsilon$.

· Consider the sequence of solutions $\{v_\varepsilon\}_\varepsilon$. Then, it is bounded in $W^{1,2}(0,1)$, such as the sequence $\{a_\varepsilon\}_\varepsilon$ is bounded in $L^2(0,1)$, where $a_\varepsilon(x) \doteq a\left(\frac{x}{\varepsilon}\right)$.

Then, there exist $v \in W^{1,2}(0,1)$ and $\bar{a} \in L^2(0,1)$ such that:

1) $v_\varepsilon \longrightarrow v$ weakly in $W^{1,2}(0,1)$ (and strongly in $L^2(0,1)$ by Rellich th.)
2) $v_\varepsilon' \longrightarrow v'$ weakly in $L^2(0,1)$
3) $a_\varepsilon \longrightarrow \bar{a}$ weakly in $L^2(0,1)$ (weakly-$*$ in $L^\infty(0,1)$), where $\bar{a} = \int_0^1 a(x)\,dx$.

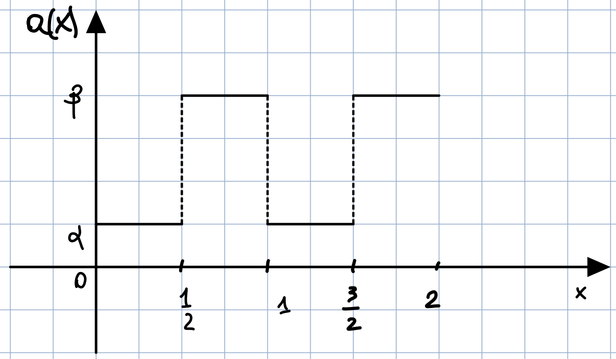
Problem: Note that, in general, $a_\varepsilon v_\varepsilon' \not\longrightarrow \bar{a} v'$ weakly in $L^2(0,1)$.

Counterexample: Let us provide an explicit formulation for a, as follows: let

$$f : (0,1) \longrightarrow \mathbb{R}$$

$$x \longmapsto \begin{cases} \alpha & \text{, if } x \in \left(0,\frac{1}{2}\right) \\ \beta & \text{, if } x \in \left(\frac{1}{2},1\right) \end{cases} \quad , \quad 0<\alpha<\beta<+\infty, \quad \alpha,\beta\in\mathbb{R}$$

and denote $a : \mathbb{R} \longrightarrow \mathbb{R}$ the extension of f to $\mathbb{R}$ by periodicity

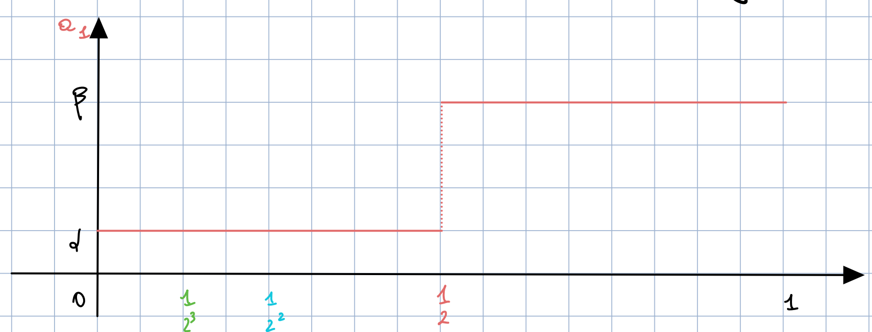


· For simplicity let $k \in \mathbb{N}$ s.t. $\varepsilon_k \longrightarrow 0$ as $k \longrightarrow +\infty$, denote

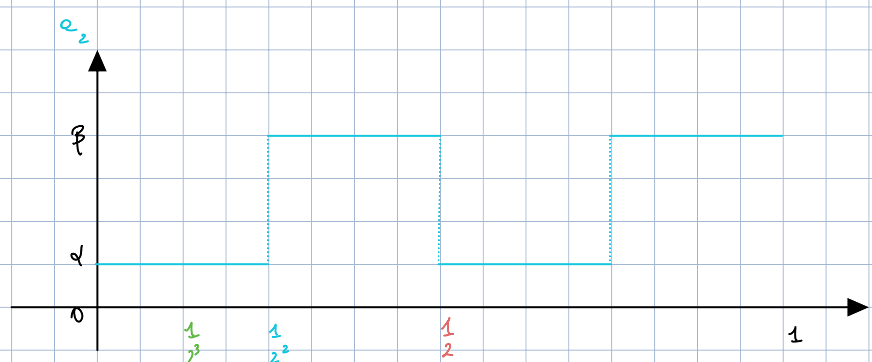
$$a_k : \left(0, \frac{1}{2^{k-1}}\right) \longrightarrow \mathbb{R}$$

$$x \longmapsto a_k(x) \doteq a(k x) = \begin{cases} \alpha & , \text{ if } x \in \left(0, \frac{1}{2^k}\right) \\ \beta & , \text{ if } x \in \left(\frac{1}{2^k}, \frac{1}{2^{k-1}}\right) \end{cases}, \quad k = 1, 2, \ldots$$

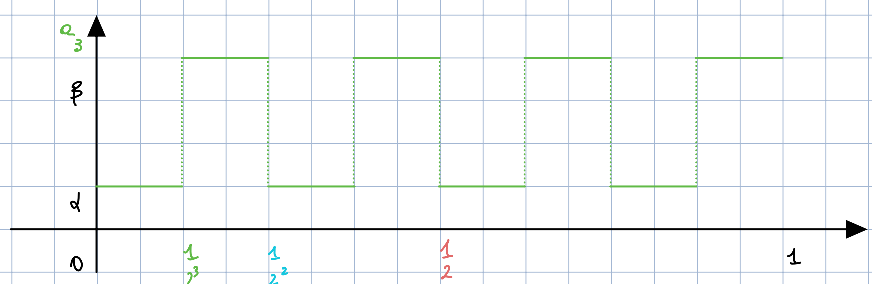
and extend a_k to $(0,1)$ by periodicity $\forall k \in \mathbb{N}$



$k = 1$



$k = 2$



$k = 3$

Then, our problem become

$$(P_h)\begin{cases} \left(a_h(x)\, v'(x)\right)' = 0 & \text{in } (0,1) \\ v(0)=0 \\ v(1)=1 \end{cases}$$

and so the solution v_h satisfies

$$\int_0^1 \left(a_h(x)\, v_h'(x)\right)' \varphi(x) = 0 \qquad \forall \varphi \in C_c^1(0,1)$$

Fund. l. c. v.

$$\Rightarrow \qquad a_h(x)\, v_h'(x) = \bar{c} \;(\in \mathbb{R}) \qquad \Rightarrow \text{the "momentum" is constant } \forall h \in \mathbb{N}.$$

If, for instance, $h=1$, we get the existence of $c_1, c_2 \in \mathbb{R}$ s.t.

$$v_1'(x) = \begin{cases} \dfrac{\bar{c}}{\alpha}, & x \in \left(0, \tfrac{1}{2}\right] \\[2ex] \dfrac{\bar{c}}{\beta}, & x \in \left[\tfrac{1}{2}, 1\right) \end{cases} \iff v_1(x) = \begin{cases} \dfrac{\bar{c}}{\alpha} x + c_1, & x \in \left(0, \tfrac{1}{2}\right] \\[2ex] \dfrac{\bar{c}}{\beta} x + c_2, & x \in \left[\tfrac{1}{2}, 1\right) \end{cases}$$

and since $v_h(0)=0$, $v_h(1)=1$, $v_h \in C\left(\left\{\tfrac{1}{2}\right\}\right)$, then

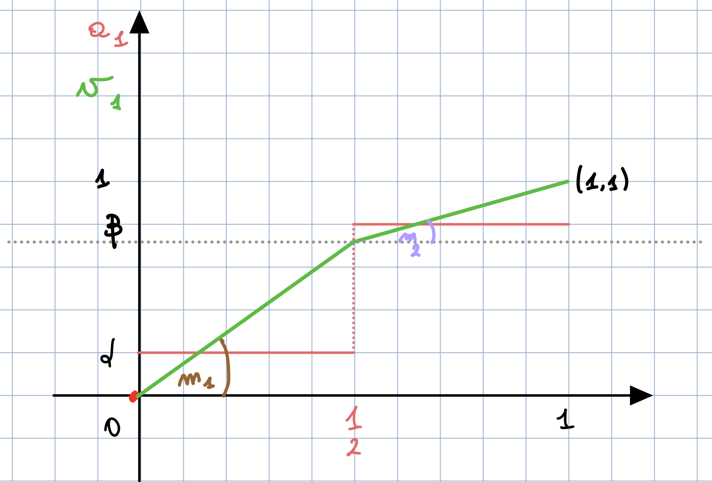
$$\begin{cases} c_1 = 0 \\[1ex] \dfrac{\bar{c}}{\beta} + c_2 = 1 \\[2ex] \dfrac{\bar{c}}{2\alpha} + c_1 = \dfrac{\bar{c}}{2\beta} + c_2 \end{cases} \iff \begin{cases} c_1 = 0 \\[1ex] c_2 = 1 - \dfrac{\bar{c}}{\beta} \\[2ex] \dfrac{\bar{c}}{2\alpha} = 1 - \dfrac{\bar{c}}{2\beta} \end{cases} \iff \begin{cases} c_1 = 0 \\[1ex] c_2 = \dfrac{\beta-\alpha}{\alpha+\beta} \\[2ex] \bar{c} = \dfrac{2\alpha\beta}{\alpha+\beta} \end{cases}$$

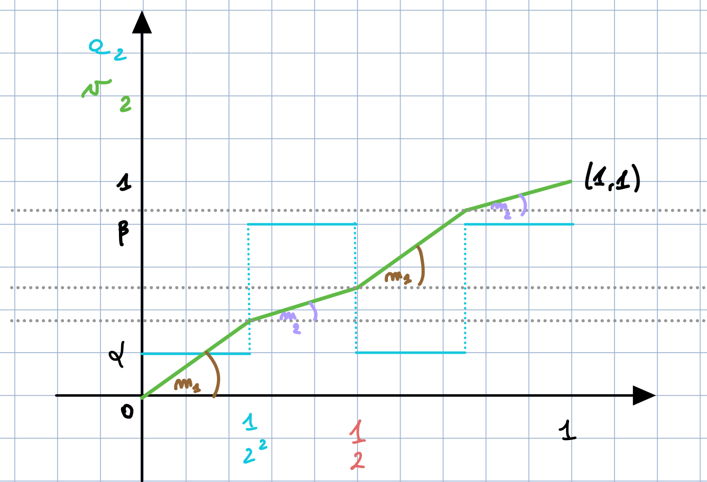
$$\iff v_1(x) = \begin{cases} \dfrac{2\beta}{\alpha+\beta} x, & x \in \left[0, \tfrac{1}{2}\right] \\[2ex] \dfrac{2\alpha}{\alpha+\beta} x + \dfrac{\beta-\alpha}{\alpha+\beta}, & x \in \left[\tfrac{1}{2}, 1\right) \end{cases}$$

$$\text{and} \quad v_1'(x) = \begin{cases} \dfrac{2\beta}{\alpha+\beta}, & x \in \left(0, \tfrac{1}{2}\right] \\[2ex] \dfrac{2\alpha}{\alpha+\beta}, & x \in \left[\tfrac{1}{2}, 1\right) \end{cases}.$$

In a similar way one can prove that (extending by periodicity) $\forall\, h \in \mathbb{N}\ \exists\, d_h, e_h \in \mathbb{R}_0^+$ s.t.

$$v_h(x) = \begin{cases} \dfrac{2\beta}{\alpha+\beta}\, x + d_h, & \text{if } x \in \left(0, \dfrac{1}{2^h}\right) \\[2ex] \dfrac{2\alpha}{\alpha+\beta}\, x + e_h, & \text{if } x \in \left(\dfrac{1}{2^h}, \dfrac{1}{2^{h-1}}\right) \end{cases} \quad \text{and} \quad v_h'(x) = \begin{cases} \dfrac{2\beta}{\alpha+\beta}, & \text{if } x \in \left(0, \dfrac{1}{2^h}\right) \\[2ex] \dfrac{2\alpha}{\alpha+\beta}, & \text{if } x \in \left(\dfrac{1}{2^h}, \dfrac{1}{2^{h-1}}\right) \end{cases}$$





with $m_1 = \dfrac{2\beta}{\alpha+\beta}$ and $m_2 = \dfrac{2\alpha}{\alpha+\beta}$.

Then,

$$a_h \longrightarrow \bar{a} = \int_0^1 a(x)\,dx = \frac{\alpha+\beta}{2} \quad \text{weakly in } L^2(0,1)$$

$$v_h' \longrightarrow \frac{\frac{2\alpha}{\alpha+\beta} + \frac{2\beta}{\alpha+\beta}}{2} = 1 \quad \text{weakly in } L^2(0,1) \quad \left(\Rightarrow v_h \longrightarrow x \text{ weakly in } H^1(0,1)\right)$$

N.B.

$$a_h(x)\, v_h'(x) = \begin{cases} \alpha \cdot \dfrac{2\beta}{\alpha+\beta}, & \text{if } x \in \left(0, \dfrac{1}{2^h}\right) \\[2ex] \beta \cdot \dfrac{2\alpha}{\alpha+\beta}, & \text{if } x \in \left(\dfrac{1}{2^h}, \dfrac{1}{2^{h-1}}\right) \end{cases} \text{(extended by periodicity in } (0,1)) = \frac{2\alpha\beta}{\alpha+\beta} \quad \forall x \in (0,1),\ \forall h \in \mathbb{N}$$

(constant)

But $\dfrac{2\alpha\beta}{\alpha+\beta} = a_h v_h' \not\longrightarrow \dfrac{\alpha+\beta}{2} \cdot 1 = \dfrac{\alpha+\beta}{2}$ (algebraic average) weakly in $L^2(0,1)$ as $h \longrightarrow +\infty$.

$\downarrow$

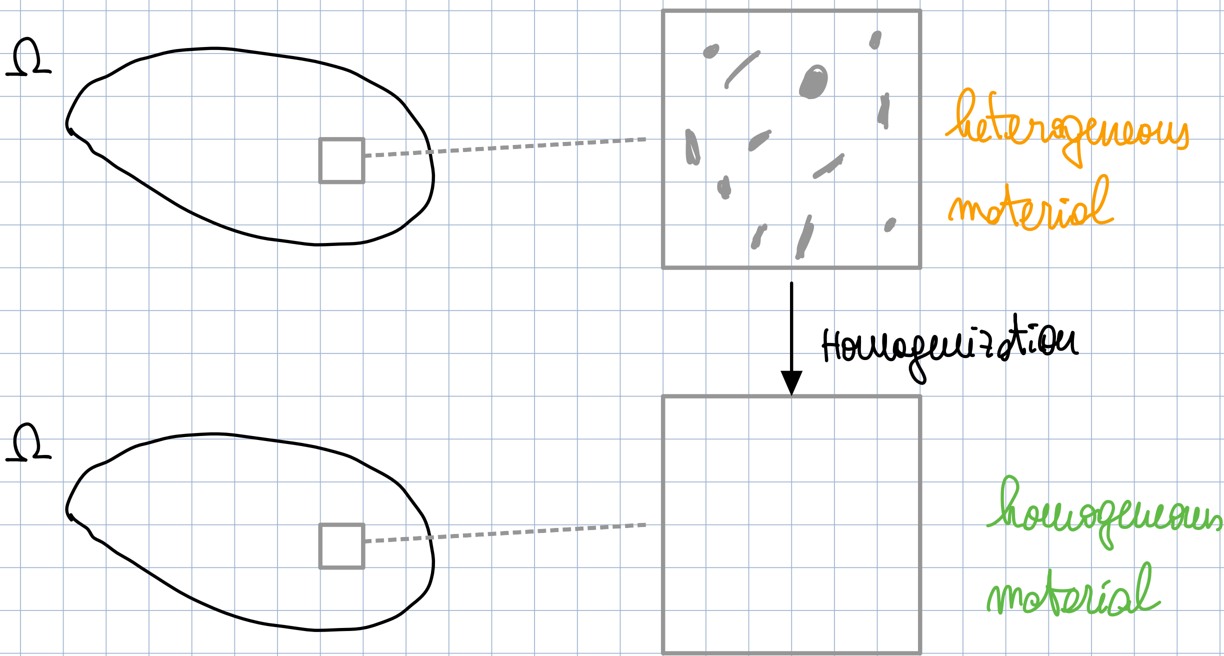
(harmonic average) $\dfrac{1}{\frac{\frac{1}{\alpha}+\frac{1}{\beta}}{2}} = \dfrac{2\alpha\beta}{\alpha+\beta}$

The mathematical theory of the homogenization of composite materials

In this last model we are interested in the study of heterogeneous materials, which are given e.g. by the superposition of sheets of different materials or by homogeneous materials with holes filled by another material.

The presence of any material inside a reference domain $\Omega \subseteq \mathbb{R}^n$, bounded domain, is expressed through a symmetric matrix-valued function $A(x) \in \mathcal{M}^{n\times n}_{sym}$, which identifies at each point $x \in \Omega$ the corresponding material.

In case of homogeneous materials, the matrix A does not depend on x.



· Mathematically speaking, this situation is modelled through elliptic PDEs in divergence form

$$-\mathrm{div}\big(\underbrace{A(x)\nabla u(x)}_{p(x) = A(x)\nabla u(x)}\big) = f(x), \quad x \in \Omega$$

$p(x) = A(x)\nabla u(x)$ momentum

$f(x)$: source term, forcing term, load term

Examples: (1) Elettrostatics: (Relevant)

$$\begin{cases} u = \text{electric potential} \\ p = \text{electric displacement} \\ A = \text{dielectric constant} \end{cases}$$

(2) Magnetostatics : $\begin{cases} u = \text{magnetic potential} \\ p = \text{magnetic induction} \\ A = \text{magnetic permeability} \end{cases}$

(3) Time-independent heat transfer : $\begin{cases} u = \text{temperature} \\ p = \text{heat flux} \\ A = \text{thermal conductivity} \end{cases}$

(4) Linear elasticity : $\begin{cases} u = \text{displacement field} \\ p (= \sigma) = \text{Cauchy stress tensor} \\ A = \text{elasticity tensor} \end{cases}$

vectorial case

$A \longrightarrow$ tensor

Remark : The assumption of symmetry in the matrix-valued function A is quite natural and covers most important applications.
However, and it goes beyond the purpose of these notes, a mathematical theory for non-symmetric models has been settled. Differently from the "symmetric" one, its correspondence with Γ-convergence has been only very recently established (2013).
The most relevant non-symmetric models are :

a) Anisotropic materials : wood, non-isotropic viscous fluids
b) Active matter models: particle dynamics in active matter systems
c) Non-symmetric elastoplasticity : materials with anisotropic magnetic-electric properties
d) Non-symmetric Schrödinger Equations in Spin Systems

The G-convergence problem

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and, fixed $0 < \lambda \leq \Lambda < +\infty$, denote

$$E(\Omega) = \Big\{ A = [a_{ij}]_{i,j \in \{1,\dots,n\}} \in L^\infty(\Omega; \mathbb{R}^{n\times n}_{\text{sym}}) : \lambda |\xi|^2 \leq \underbrace{A(x)\xi\cdot\xi}_{f(x,\xi)} \leq \Lambda |\xi|^2$$
$$\text{for all } \xi \in \mathbb{R}^n \text{ for a.e. } x \in \Omega \Big\}.$$

For any matrix $A \in E(\Omega)$ consider the differential operators

$$\mathcal{A} : H^1_0(\Omega) \longrightarrow H^{-1}(\Omega)$$
$$u \longmapsto \mathcal{A}u \doteq -\operatorname{div}(A(x)\nabla u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}u\Big)$$

that is

$$\mathcal{A}u : H^1_0(\Omega) \longrightarrow \mathbb{R}$$
$$v \longmapsto \int_\Omega A(x)\nabla u \cdot \nabla v \, dx.$$

· Fix now a sequence of matrix-valued functions $\{A_h\}_h \subseteq E(\Omega)$ & consider the sequence of problems

$$P_h \begin{cases} -\operatorname{div}(A_h(x)\nabla u_h) = f & , \text{ in } \Omega \\ u_h = 0 & , \text{ in } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$ is fixed (by density we may assume $f \in H^{-1}(\Omega)$) and $h \in \mathbb{N}$.

· We note that the natural space of solutions for any problem P_h is $H^1_0(\Omega)$ and that each problem P_h is well-defined, according with the following result.

__Lemma__: For any $f \in H^{-1}(\Omega)$ $\forall h \in \mathbb{N}$ there exits a unique weak solution of P_h, that is $\exists\, \overline{u_h} \in H^1_0(\Omega)$ s.t.

$$\int_\Omega A_h(x)\nabla \overline{u}_h \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H^1_0(\Omega).$$

Remark: • The proof of the lemma is a direct consequence of Lax-Milgram lemma.
In fact, the bilinear form

$$a : H^1_0(\Omega) \times H^1_0(\Omega) \longrightarrow \mathbb{R}$$
$$(u, v) \longmapsto \int_\Omega A_h(x) \nabla u \cdot \nabla v \, dx$$

is coercive and continuous, whenever $A_h \in E(\Omega)$.

• As a consequence, any operator $\mathcal{A}_h$ is invertible.

• The G-convergence problem consists in finding a limit matrix $A_\infty \in E(\Omega)$ s.t. if we consider the limit problem

$$P_\infty \begin{cases} -\mathrm{div}(A_\infty(x) \nabla u_\infty) = f & , \text{ in } \Omega \\ u_\infty = 0 & , \text{ in } \mathbb{R}^n \setminus \Omega \end{cases} \qquad (\mathcal{A}_\infty u_\infty = f)$$

(for the same function f), then

$$u_h \longrightarrow u_\infty \quad \text{weakly in } H^1_0(\Omega)$$
$$\text{strongly in } L^2(\Omega) \quad (\text{by Rellich Theorem}).$$

Def: Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and let $A_h, A_\infty \in E(\Omega)$, for fixed (positive) constants $\lambda \leq \Lambda$.
We say that $\{\mathcal{A}_h\}_h$ G-converges to $\mathcal{A}_\infty$ if

$$\mathcal{A}_h^{-1} f \longrightarrow \mathcal{A}_\infty^{-1} f \quad \text{strongly in } L^2(\Omega) \text{ as } h \to 0 \ \forall f \in L^2(\Omega)$$
$$\Downarrow$$
$$(\forall f \in H^{-1}(\Omega))$$

This problem was solved by De Giorgi and Spagnolo in the late '60s and, few years later, by Tartar and Murat in the case of (possibly) non-symmetric matrices. In this case we talk about H-convergence.

This last case is much delicate, because the divergence operator does not recognize the presence of skew-symmetric matrices, and we may have a "lack of uniqueness" in the limit operator.

<u>Remark</u>: The closure of the class $E(\Omega)$ under the G-convergence, also known as G-compactness, is proved with "operatorial techniques", and is a particular case of the compensated compactness theory.

· We now want to show how to equivalently obtain such result by Γ-convergence of lower semicontinuous quadratic forms.

· We first note that any problem P_h has a variational characterization: For any $h \in \mathbb{R}^+$ we denote

$$F_h : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$$

$$u \longmapsto \begin{cases} \int_\Omega A_h(x) \nabla u \cdot \nabla u \, dx, & \text{if } u \in H_0^1(\Omega) \\ +\infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega) \end{cases}$$

and

$$G : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$$

$$u \longmapsto \int_\Omega f u \, dx$$

· Then we have the following equivalence:

$$u_h \text{ is a solution of } P_h \iff u_h \in \min \{ F_h(u) - G(u) : u \in L^2(\Omega) \}$$

<u>Q</u>: What happens if we consider the sequence of weak solutions $\{u_h\}_h$?

Th: (Sbordone 1975)

Let $\{F_h\}_h$ be the sequence of energy functionals associated with $\{A_h\}_h \subseteq E(\Omega)$.
Then, there exists $A_\infty \in E(\Omega)$ s.t. (up to subsequences)

$$\{F_h\}_h \ \Gamma\text{-converges to } F_\infty \text{ in the strong topology of } L^2(\Omega),$$

where
$$F_\infty : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$$
$$u \longmapsto \begin{cases} \int_\Omega A_\infty(x) \nabla u \cdot \nabla u \, dx , & \text{if } u \in H_0^1(\Omega) \\ +\infty , & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega) \end{cases}$$

· The last goal of this notes is the following equivalent proof of the G-compactness, by Γ-convergence.

Th: Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and let $\{A_h\}_h \subseteq E(\Omega)$, for fixed (positive) constants $\lambda \le \Lambda$. Then, there exists $A_\infty \in E(\Omega)$ s.t.

$$\{\mathcal{A}_h\}_h \ G\text{-converges to } \mathcal{A}_\infty .$$

up to subsequences

Proof: (Variational)

Given any problem P_h, let F_h be the corresponding energy functional.
By the previous theorem, $\exists\, A_\infty \in E(\Omega)$ s.t. (up to subsequences)

$$\{F_h\}_h \ \Gamma\text{-converges to } F_\infty \text{ in the strong topology of } L^2(\Omega)$$

where F_∞ is defined above.

· By the properties of Γ-convergence, since G is a "continuous perturbation" of u in $L^2(\Omega)$, then

$$\{F_h - G\}_h \ \Gamma\text{-converges to } F_\infty - G$$

in the strong topology of $L^2(\Omega)$.

To conclude we have to show that the sequence $\{F_{\ell}-G\}_{\ell}$ is equi-coercive. In fact, by the Fundamental theorem of Γ-convergence, we would obtain that the sequence of minimizers $\{u_{\ell}\}_{\ell}$ for $\{F_{\ell}-G\}_{\ell}$ is s.t.:

1) $u_{\ell} \longrightarrow u_{\infty}$ weakly in $H^1_0(\Omega)$

unique minimizer of F_{∞}

2) $u_{\ell} \longrightarrow u_{\infty}$ strongly in $L^2(\Omega)$

3) $u_{\ell} = \min\limits_{H^1_0(\Omega)} F_{\ell}-G$, $u_{\infty} = \min\limits_{H^1_0(\Omega)} F-G$

4) $u_{\ell} = \min\limits_{L^2(\Omega)} F_{\ell}-G$, $u_{\infty} = \min\limits_{L^2(\Omega)} F-G$

5) u_{ℓ} solves P_{ℓ} , u_{∞} solves P_{∞}

6) $u_{\ell} = \mathcal{A}_{\ell}^{-1}(f)$. $u_{\infty} = \mathcal{A}_{\infty}^{-1}(f)$

As a consequence of (2) and (6), $\{\mathcal{A}_{\ell}\}_{\ell}$ G-converges to $\mathcal{A}_{\infty}$.

$\square$

(Proof of the equicoercivity)

Fix $\ell \in \mathbb{R}^+$. Assume that

$$+\infty > F_{\ell}(u_{\ell}) - G(u_{\ell}) = \int_{\Omega} A_{\ell}(x) \nabla u_{\ell} \cdot \nabla u_{\ell} \, dx - \int_{\Omega} f u_{\ell} \, dx$$

$$\overset{A_{\ell} \in E(\Omega)}{\geq} \lambda \int_{\Omega} |\nabla u_{\ell}|^2 \, dx - \int_{\Omega} f u_{\ell} \, dx$$

$$\overset{f \in L^2(\Omega)}{\geq} c \, \|u_{\ell}\|^2_{H^1_0(\Omega)} .$$

Then, $\{u_{\ell}\}_{\ell}$ is a bounded sequence in the space $H^1_0(\Omega)$ and the thesis follows by the reflexivity of the spaces and Rellich Theorem.

$\square$