03.12.2024

$\boxed{\text{STEP 2}}$: Equi-coercivity of $\{F_\varepsilon\}_\varepsilon$.

<u>CLAIM</u>: $F_\varepsilon(u_\varepsilon) < C_1 \ \forall \varepsilon \in \mathbb{R}^+ \Rightarrow \exists \ \varepsilon_j \searrow 0$ as $j \nearrow +\infty$ $\exists \ u \in X$ s.t. $u_{\varepsilon_j} \longrightarrow u$ in $L^1(a,b)$-strong.

· Fix $u_\varepsilon \in X$ s.t. $F_\varepsilon(u_\varepsilon) < \infty$. Then, $u_\varepsilon \in W^{1,2}(a,b)$ and $u_\varepsilon \in [0,1]$.

Consider the map
$$\phi : [0,1] \longrightarrow \mathbb{R} \cup \{+\infty\}$$
$$t \longmapsto \phi(t) \doteq \int_0^t \sqrt{W(s)}\, ds$$

By construction $\phi \in C^1(0,t)$ (W is continuous) and ϕ is strictly increasing, being $\phi'(t) > 0 \ \forall t \in [0,1]$).

Moreover, denote $v_\varepsilon(t) \doteq \phi(u_\varepsilon(t)) \Rightarrow v_\varepsilon \in W^{1,2}(a,b)$, $\|v_\varepsilon\|_{L^\infty} \le C_1$ and, by the chain rule,
$$v_\varepsilon'(t) = \phi'(u_\varepsilon(t))\, u_\varepsilon'(t) = \sqrt{W(u_\varepsilon(t))} \cdot u_\varepsilon'(t)$$
and so
$$\int_a^b |v_\varepsilon'(t)|\, dt = \int_a^b \sqrt{W(u_\varepsilon(t))} \cdot |u_\varepsilon'(t)|\, dt$$
$$\underset{\text{Young's inequ.}}{\le} \frac{1}{2\varepsilon}\int_a^b W(u_\varepsilon(t))\, dt + \frac{\varepsilon}{2}\int_a^b |u_\varepsilon'(t)|^2\, dt$$
$$= \frac{1}{2} F_\varepsilon(u_\varepsilon) \le C_1 \Rightarrow \|v_\varepsilon\|_{W^{1,1}(a,b)} \le C_1 \qquad \forall \varepsilon \in \mathbb{R}^+$$

Then, $\{v_\varepsilon\}_\varepsilon$ is uniformly bounded in $W^{1,1}(a,b)$ and so $\exists \ \varepsilon_j \searrow 0$ as $j \nearrow +\infty$ and $\exists \ v \in L^1(a,b)$ s.t. $\quad v_{\varepsilon_j} \longrightarrow v$ in $L^1(a,b)$.

To conclude, denote $u \doteq \phi^{-1}(v)$. Then, by the dominated convergence theorem
$$u_{\varepsilon_j} = \phi^{-1}(v_{\varepsilon_j}) \longrightarrow \phi^{-1}(v) = u \quad \text{in } L^1(a,b)\text{-strong}$$

which implies the equicoercivity of $\{F_\varepsilon\}_\varepsilon$.

$\boxed{\text{STEP 3}}$: (Γ- limsup - inequality)

Fix $u \in X$. We aim to find a recovery sequence $u_\varepsilon \longrightarrow u$ in $L^1(a,b)$ s.t.

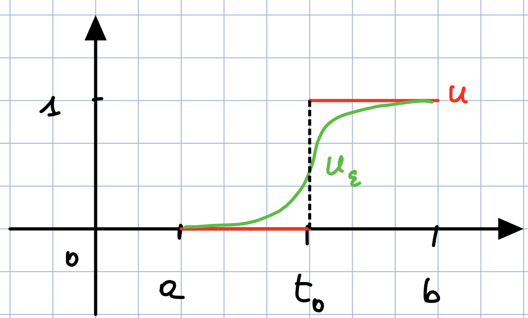
$$\limsup_{\varepsilon \searrow 0} F_\varepsilon(u_\varepsilon) \leq F(u).$$

Assume $F(u) < +\infty$, otherwise the thesis is trivial.

Then $u \in BV((a,b);\{0,1\})$ and $S_u \neq \emptyset$ (otherwise $u \equiv 0$ or $u \equiv 1$ and $\int_a^b u(x)dx = V$ is not satisfied). It means that there is (at least) one jump.

· We first assume that there is only one jump $\Rightarrow \exists\, t_0 \in (a,b)$ s.t.

$$\lim_{t \to t_0^-} u(t) \neq \lim_{t \to t_0^+} u(t)$$



We want u_ε to create a smooth transition between $u=0$ and $u=1$.

<u>Remark</u>: $\displaystyle\int_{t_0-\delta}^{t_0+\delta} \left[\frac{1}{\varepsilon} W(u_\varepsilon(x)) + \varepsilon\, |u_\varepsilon'(x)|^2 \right] dx \geq 2\int_{t_0-\delta}^{t_0+\delta} \sqrt{W(u_\varepsilon(x))}\, |u_\varepsilon'(x)|\, dx \geq \sigma.$

· Consider the following minimization problem "optimal profile problem" (in all $\mathbb{R}$):

$$\overline{\sigma} \doteq \min\left\{ \int_{-\infty}^{+\infty} \left[W(v(x)) + |v'(x)|^2 \right] dx : v \in W^{1,2}_{loc}(\mathbb{R}),\ v(-\infty) = 0,\ v(+\infty) = 1 \right\}$$

<u>CLAIM</u>: $\overline{\sigma}$ exists (meaning that the minimum is reached). Moreover, $\overline{\sigma} = \sigma = 2\displaystyle\int_0^1 \sqrt{W(\tau)}\, d\tau.$

<u>Proof (CLAIM)</u>: Assume v admissible. Then,

$$\int_{-\infty}^{+\infty} \left[W(v(x)) + |v'(x)|^2 \right] dx \geq 2\int_{-\infty}^{+\infty} \sqrt{W(v(x))} \cdot |v'(x)|\, dx \geq 2\int_{\underset{v(-\infty)}{\overset{\shortparallel}{0}}}^{\overset{v(+\infty)}{\overset{\shortparallel}{1}}} \sqrt{W(\tau)}\, d\tau = \sigma$$

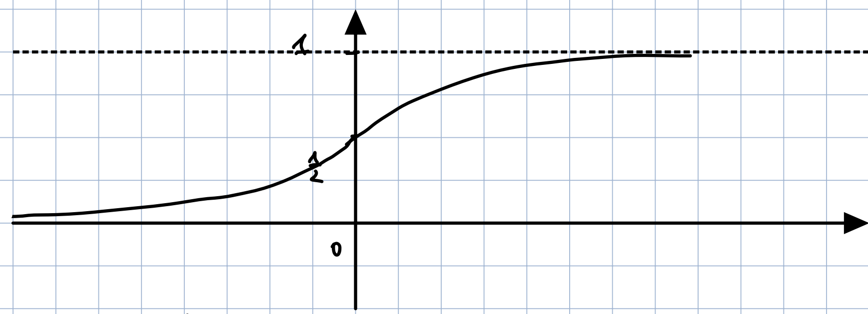
and so $\overline{\sigma} \geq \sigma$. Let us show that $\overline{\sigma} \leq \sigma$.

Notice that the only way to obtain the claim is to show that

$$\cancel{\alpha^2 + \beta^2 \geq 2\alpha\beta} \qquad \alpha^2 + \beta^2 = 2\alpha\beta$$

Consider the Cauchy problem

$$\begin{cases} v'(x) = \sqrt{W(v(x))}, & x \in \mathbb{R} \\ v(0) = \frac{1}{2} \end{cases}$$



$v = 0$ and $v = 1$ are stationary solutions of the (ODE)

$\sqrt{W(\cdot)}$ is continuous $\Rightarrow \exists!$ solution $\Rightarrow \exists\, \bar{v} : \mathbb{R} \to (0,1)$ sol. of the (ODE)

$\bar{v}' = \sqrt{W(\bar{v})}$ continuous and positive $\Rightarrow \bar{v}$ (strictly) increasing in $\mathbb{R}$ and such that

$$\lim_{x \to -\infty} v(x) = 0, \quad \lim_{x \to +\infty} v(x) = 1.$$

Then, the solution $\bar{v}$ is an admissible profile for the problem, and so

$$\overset{\min}{\overline{\sigma}} \leq \int_{-\infty}^{+\infty} \left[W(\bar{v}(x)) + |\bar{v}'(x)|^2 \right] dx \underset{\substack{\bar{v} \\ \text{sol.}}}{=} 2 \int_{-\infty}^{+\infty} \sqrt{W(\bar{v}(x))}\, \bar{v}'(x)\, dx \overset{\text{change of var.}}{=} \sigma$$

$\Rightarrow \bar{\sigma} = \sigma$ and $\bar{v}$ is an optimal profile. $\square$

- Let us build by hand a recovery sequence for $u(x) = \begin{cases} 0 & \text{if } x \in (a, t_0) \\ 1 & \text{if } x \in (t_0, b) \end{cases}$

As a consequence of the claim

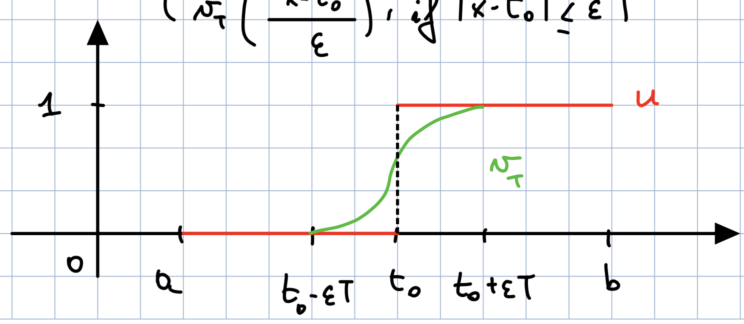
$$\sigma = \min \left\{ \int_{-\infty}^{+\infty} \left[W(v(x)) + |v'(x)|^2 \right] dx : v \in W^{1,2}_{loc}(\mathbb{R}),\ v(-\infty) = 0,\ v(+\infty) = 1 \right\}$$

$$= \inf_{T \nearrow +\infty} \inf \left\{ \int_{-T}^{+T} \left[W(v(x)) + |v'(x)|^2 \right] dx : v \in W^{1,2}(-T,T),\ v(-T) = 0,\ v(+T) = 1 \right\}.$$

For any fixed $\eta \in \mathbb{R}^+$ $\exists\, T \in \mathbb{R}^+$ and $v_T \in W^{1,2}(-T,T)$ such that $v_T(-T) = 0$, $v_T(+T) = 1$ and $\int_{-T}^{+T} \left[W(v_T(x)) + |v_T'(x)|^2 \right] dx < \sigma + \eta$. (infimum)

We then define

$$u_\varepsilon(x) = \begin{cases} u(x) & , \text{if } |x - t_0| > \varepsilon T \\ v_T\left(\dfrac{x - t_0}{\varepsilon}\right), & \text{if } |x - t_0| \leq \varepsilon T \end{cases}$$



By construction,

i) $u_\varepsilon \in W^{1,2}(a,b)$ and $u_\varepsilon(x) \in [0,1]$ $\forall x \in (a,b)$

ii) $u_\varepsilon(t_0+\varepsilon T) = v_T(T) = 1$

iii) $u_\varepsilon(t_0-\varepsilon T) = v_T(-T) = 0$

iv) $u_\varepsilon \longrightarrow u$ in $L^1(a,b)$

v) $u_\varepsilon'(x) = \begin{cases} 0 & , \text{if } |x-t_0| > \varepsilon T \\ \frac{1}{\varepsilon} v_T'\left(\frac{x-t_0}{\varepsilon}\right) & , \text{if } |x-t_0| \leq \varepsilon T \end{cases}$

vi)
$$F_\varepsilon(u_\varepsilon) = \int_a^b \left[\frac{1}{\varepsilon} W(u_\varepsilon(x)) + \varepsilon\, |u_\varepsilon'(x)|^2 \right] dx$$
$$= \int_{t_0-\varepsilon T}^{t_0+\varepsilon T} \left[\frac{1}{\varepsilon} W\left(v_T\left(\frac{x-t_0}{\varepsilon}\right)\right) + \cancel{\varepsilon} \cdot \frac{1}{\varepsilon^{\cancel{2}}} \left| v_T'\left(\frac{x-t_0}{\varepsilon}\right)\right|^2 \right] dx$$

C.V. $y = \frac{x-t_0}{\varepsilon}$

$$= \int_{-T}^{T} \left[\frac{1}{\cancel{\varepsilon}} W(v_T(y)) + \frac{1}{\cancel{\varepsilon}} |v_T'(y)|^2 \right] \cancel{\varepsilon}\, dy \leq \sigma + \eta = F(u) + \eta$$

$1 = \mathcal{H}^0\{t_0\}$

$F(u) = \sigma \cdot \mathcal{H}^0(S_u) = \sigma$

• Fix now a sequence $\{\eta_\varepsilon\}_\varepsilon \subseteq \mathbb{R}^+$ s.t. $\eta_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$ and, in a similar way, build $\{u_\varepsilon\}_\varepsilon$. Then, by (vi)

$$\limsup_{\varepsilon \searrow 0} F_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon \searrow 0} \left[F(u) + \eta_\varepsilon \right] = F(u)$$

• To conclude we show that any $u_\varepsilon \in X$, meaning that

$$\int_a^b u_\varepsilon(x)\, dx = V.$$

To this aim, we extend u_ε outside (a,b), by considering (with abuse of notation)

$$u_\varepsilon : \mathbb{R} \longrightarrow [0,1]$$
$$x \longmapsto \begin{cases} 0 & , \text{if } x < a \\ u_\varepsilon(x) & , \text{if } x \in [a,b] \\ 1 & , \text{if } x > b \end{cases}$$

Then, $V_\varepsilon := \int_a^b u_\varepsilon(x)\, dx \longrightarrow \int_a^b u(x) dx = V$ as $\varepsilon \searrow 0$.

Let now $\tilde{u}_\varepsilon : \mathbb{R} \longrightarrow [0,1]$

$$x \longmapsto \tilde{u}_\varepsilon(x) \doteq u_\varepsilon(x - x_\varepsilon)$$

where $x_\varepsilon \doteq V_\varepsilon - V$ ($x_\varepsilon \longrightarrow 0$ as $\varepsilon \searrow 0$ and w.l.o.g. $x_\varepsilon > 0$). Then,

$$\begin{aligned}
\int_a^b \tilde{u}_\varepsilon(x)\,dx &= \int_a^b u_\varepsilon(x - x_\varepsilon)\,dx \\
\text{c.v. } y = x - x_\varepsilon &= \int_{a-x_\varepsilon}^{b-x_\varepsilon} u_\varepsilon(y)\,dy \\
&= \cancel{\int_{a-x_\varepsilon}^{a} u_\varepsilon(y)\,dy} + \int_a^b u_\varepsilon(y)\,dy - \int_{b-x_\varepsilon}^{b} \overbrace{u_\varepsilon(y)}^{1}\,dy \\
&= \int_a^b u_\varepsilon(y)\,dy - x_\varepsilon = V_\varepsilon - x_\varepsilon = V.
\end{aligned}$$

Therefore, $\tilde{u}_\varepsilon \in X$ $\forall \varepsilon \in \mathbb{R}^+$, $\tilde{u}_\varepsilon \longrightarrow u_1$ in $L^1(a,b)$-str. and

$$\limsup_{\varepsilon \searrow 0} F_\varepsilon(\tilde{u}_\varepsilon) \leq F(u).$$

We can repeat the same construction for any (finite) set of jumps.

$\square$

Corollary: Let $\Omega, X, F_\varepsilon, F, E_\varepsilon$ be such as in the Modica-Mortola Theorem.

Then, every sequence of minimizers $\{u_\varepsilon\}_\varepsilon$ of $\{F_\varepsilon\}_\varepsilon$, $\varepsilon \in \mathbb{R}^+$, is precompact in X and every limit $u \in BV(\Omega; \{0,1\}) \cap X$ minimizes E.

or $\{E_\varepsilon\}_\varepsilon$ or $\{\lambda_\varepsilon E_\varepsilon\}_\varepsilon$

Remark: If the model presents more than two phases (n-phases, $n>2$), the corresponding potential energy is not, in general, governed by the n^{th}-well potential.

There are "more convenient" energy densities, depending on each model.