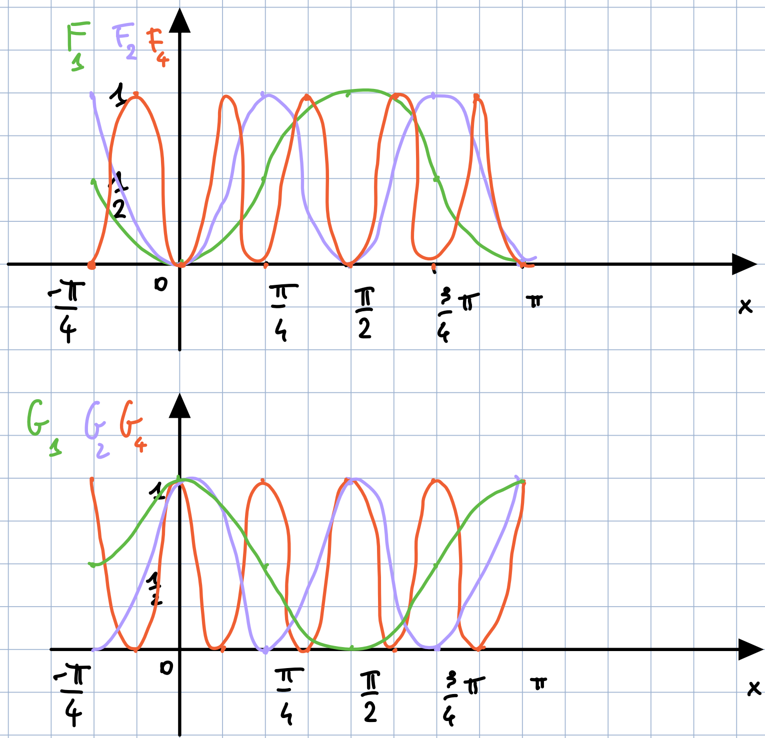
Example: Let $X = \mathbb{R}$ and let $F_J : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto \sin^2(Jx)$ and 07.11.2024

$G_J : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto \cos^2(Jx)$ for any $J \in \mathbb{N}$.



We want to find a Γ-limit "candidate" for both sequences $\{F_J\}_J$ and $\{G_J\}_J$ (we work only with the first sequence, being the other case analogous).

· Let us fix an arbitrary point $\bar{x} \in \mathbb{R}$ and look for a lower bound for $\{F_J\}_J$ in a neighbourhood of $\bar{x}$.

It can be, for instance, any costant function less or equal than 0 (in this way condition (i) is satisfied).

To ensure the validity of condition (ii) (or (ii')) we take the greatest lower bound for $\{F_J\}_J$, i.e., $F = 0$.

· Let us then show that conditions (i) and (ii') are satisfied.

(i) trivial

(ii) We have to find a recovery sequence $\{x_J\}_J$ for any fixed point $\bar{x} \in \mathbb{R}$.

Let $x_J = \left\lceil \frac{J}{\pi} \bar{x} \right\rceil \cdot \frac{\pi}{J}$ for any $J \in \mathbb{N}$, where $\lceil x \rceil$ = smallest integer $\geq x$.

Then $x_J = \begin{cases} \bar{x}, & \text{if } \frac{J}{\pi}\bar{x} \in \mathbb{Z} \\ \geq \bar{x}, & \text{otherwise} \end{cases}$ and, by density, $x_J \longrightarrow \bar{x}$ as $J \longrightarrow +\infty$

and $F_J(x_J) = \sin^2(J x_J) = \sin^2(k\pi) = 0 = F(\bar{x})$ $(\forall \bar{x} \in \mathbb{R})$ $\forall J \in \mathbb{N}$.

· Consider now $H_J(x) \doteq F_J(x) + G_J(x) = 1$ $\forall x \in \mathbb{R}$ $\forall J \in \mathbb{N}$ and observe that

$$(F_J + G_J =) \, H_J \xrightarrow{\Gamma} H = 1 \neq 0 = F + G$$

<u>Remark</u>: Without further assumptions, if $\{F_J\}_J$ and $\{G_J\}_J$ Γ-converge to F and G, then

$$\boxed{\Gamma\text{-}\lim_{J \to +\infty} (F_J + G_J) \neq \Gamma\text{-}\lim_{J \to +\infty} F_J + \Gamma\text{-}\lim_{J \to +\infty} G_J .}$$

A case in which the Γ-limit of a sum is the sum of the Γ-limits is:

<u>Prop</u>: Assume that:

i) $F_J \xrightarrow{\Gamma} F$ in X;

ii) $(G_J)_J$ continuously converges to G in X, meaning that

$\forall u \in X$ $\forall V$ neighbourhood of $G(u)$ in $\overline{\mathbb{R}}$ $\exists N \in \mathbb{N}$ $\exists U \in \mathcal{N}(u)$ s.t.

$$G_J(v) \in V \quad \forall J \geq N \ \forall v \in U$$

iii) G_J, G are finite everywhere in X.

Then, $F_J + G_J \xrightarrow{\Gamma} F + G$ in X.

<u>Remark</u>: The continuous convergence is stronger than the pointwise, while the uniform convergence implies the continuous, if the limit is continuous.

· There is in any case "stability" w.r.t. continuous perturbations.

<u>Th</u>: Let (X, d) be a metric space and let $F_J, F, G : X \longrightarrow \overline{\mathbb{R}}$ satisfy:

i) $F_J \xrightarrow{\Gamma} F$

ii) G is continuous in X (w.r.t. the metric d).

Then, $F_J + G \xrightarrow{\Gamma} F + G$ (provided $F + G$ is well-defined).

<u>Ex</u>: Prove the previous results.

3) A comparison of convergences (in this section we always assume X equipped with a metric d)

"All the other convergences can be expressed in the language of Γ-convergence."

Def: (Pointwise convergence) Let $F_j, F : X \longrightarrow \overline{\mathbb{R}}$ and fix $\bar{x} \in X$. We say that $\{F_j\}_j$ pointwise converges to F in $\bar{x}$ if $\lim_{j \to +\infty} F_j(\bar{x}) = F(\bar{x})$, i.e.

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $|F_j(\bar{x}) - F(\bar{x})| < \varepsilon \quad \forall j \geq N$.

We then say that $\{F_j\}_j$ pointwise converges to F if it happens $\forall \bar{x} \in \mathbb{R}$.

Def: (Uniform convergence) We say that $\{F_j\}_j$ uniformly converges to F in any subset $I \subseteq X$ ($F_j \overset{I}{\rightrightarrows} F$ in short) if the sequence $\{\sup_I |F_j - F|\}_j$ converges to 0, i.e.

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $|F_j(x) - F(x)| < \varepsilon \quad \forall j \geq N \ \forall x \in I$.

Remark: If $F_j \overset{I}{\rightrightarrows} F$, then $\{F_j\}_j$ pointwise converges to F in I.

Q: How does Γ-convergence behave w.r.t. pointwise and uniform convergence?

Example: Let $F : X \longrightarrow \overline{\mathbb{R}}$ be any arbitrary sequence and let $F_j = F \ \forall j \in \mathbb{N}$.

Observe that, if F is not lower semicontinuous, then

$$F_j = F \overset{\Gamma}{\not\longrightarrow} F.$$

In fact, condition (i) fails: fix any $\bar{x} \in X$ and $x_j \longrightarrow \bar{x}$ in X. Then

$$F_j \xrightarrow{\Gamma} F \quad \Rightarrow F(x) \overset{(i)}{\leq} \liminf_{j \to +\infty} F_j(x_j) = \liminf_{j \to +\infty} F(x)$$

Prop: $F_j = F \xrightarrow{\Gamma} F$ if and only if F is lower semicontinuous.

Ex: Prove the proposition.

· The previous proposition tells that Γ-convergence and pointwise convergence are not comparable.

Prop: Fix I open subset of X and let $F_j \overset{I}{\rightrightarrows} F$. If F is lower semicontinuous then $F_j \xrightarrow{\Gamma} F$ in I.

Proof: (i) Fix $u \in I$ and $u_j \longrightarrow u$ in X. Since I is open, $u_j \in I$ $\forall j$ big enough and $|F_j(u_j) - F(u_j)| \leq \sup_I |F_j - F| \longrightarrow 0$ by hypothesis.

Therefore,
$$\liminf_{j \to +\infty} F_j(u_j) = \liminf_{j \to +\infty} F(u_j) \overset{\text{l.s.c.}}{\geq} F(u).$$

(ii) $\forall u \in I$ let $u_j = u$ $\forall j \in \mathbb{N}$. Then, $\lim_{j \to +\infty} F_j(u_j) = F(u)$. $\square$

Remark: The lower semicontinuity assumption is not accidental. We will see that any Γ-limit F is always a lower semicontinuous functional (in some sense it was build for attend minima and so this is a consequence of the Direct Methods in Calculus of variations).

4) Fundamental Theorem of Γ-convergence

We now provide the most important result concerning Γ-convergence.

Th: Fundamental theorem of Γ-convergence – De Giorgi, Franzoni (1975)

Let (X, d) be a metric space and let $F_j, F : X \longrightarrow \overline{\mathbb{R}}$ satisfy:

i) $F_j \xrightarrow{\Gamma} F$ $\qquad$ F_j coercive $\Rightarrow$ F_j mildly coercive

ii) $\{F_j\}_j$ is equi-mildly coercive, i.e. $\exists$ $K \subseteq X$ compact, $K \neq \emptyset$ s.t.
$$\inf_X F_j = \inf_K F_j \quad \forall j \in \mathbb{N}.$$
K is independent of j

weaker formulation

Then: 1) there exists $\min_X F$

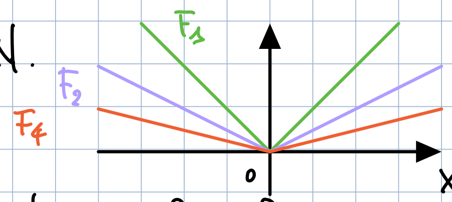
2) there exists $\lim_{j \to +\infty} \inf_X F_j = \min_X F$

3) if $\{u_j\}_j \subseteq X$ is a relatively compact sequence satisfying
$$\lim_{j \to +\infty} F_j(u_j) = \lim_{j \to +\infty} \inf_X F_j,$$
(strongly) convergent up to subsequences

then, $\lim_{j \to +\infty} u_j \in \{u \in X : F(u) = \min_X F\}$.

Remark: As a consequence of the fundamental theorem of Γ-convergence, if $F_j(u_j) = \min_X F_j$ $\forall j \in \mathbb{N}$ and $\exists\, u \in X$ s.t. $u_j \to u$ in X, then $F(u) = \min_X F$. Unfortunately the viceversa is wrong.

Example: Let $X = \mathbb{R}$ and let $F_j(x) \doteq \frac{|x|}{j}$ $\forall j \in \mathbb{N}$.



One can easily show that

$F_j \xrightarrow{\Gamma} F = 0$, which has an infinite number of minimizers.

If we consider any minimizer for F different from $x = 0$, we cannot find a sequence $x_j \to x \neq 0$ satisfying

$$F_j(x_j) = \min_{\mathbb{R}} F_j \quad \forall j \in \mathbb{N}.$$

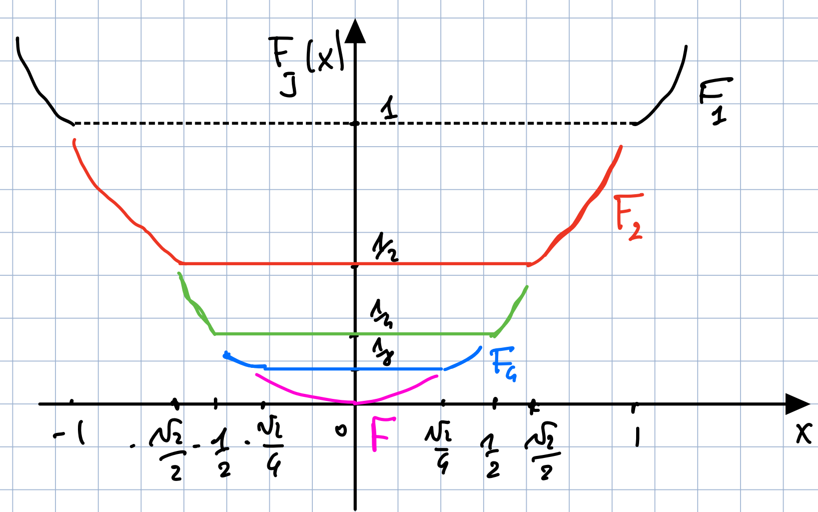
$\Rightarrow$ $\boxed{\text{The } \Gamma\text{-convergence acts as a "selection process" of minima.}}$

Remark: We may wonder whether, under appropriate assumptions, a "reverse" statement of the Fundamental Theorem of Γ-convergence may be possible, in the following sense:

Q: if the Γ-limit F attains only one unique minimizer, then exists $N \in \mathbb{N}$ (big enough) s.t. F_j also has only one minimizer $\forall j \geq N$?

Counterexample: Let $X = \mathbb{R}$ and let $F_j(x) \doteq \max\left\{x^2, \frac{1}{j}\right\}$ for any $j \in \mathbb{N}$.

Then, $F_j \xrightarrow{\Gamma} F(x) = x^2$, but



<u>Proof</u> (Fundamental th.):

Since $\{F_j\}_j$ is equi-mildly coercive, there exists a sequence $\{u_j\}_j \subseteq K$ s.t.

$$\liminf_{j \to +\infty} F_j(u_j) = \liminf_{j \to +\infty} \inf_X F_j$$

(we work with the liminf because, in general, $\lim_{j \to +\infty} F_j(u_j)$ does not exist).

By the compactness of K, there exists a subsequence $\{u_{j_k}\}_k \subseteq \{u_j\}_j$ s.t.

$$\lim_{k \to +\infty} F_{j_k}(u_{j_k}) = \liminf_{j \to +\infty} F_j(u_j)$$

(this is always true, being the liminf of a subsequence greater or equal to the liminf of the entire sequence. Moreover, there is always a subseq. whose liminf is a limit).

By hypotheses, up to passing to a further (not relabelled) subsequence, there exists $\bar{u} \in K$ s.t. $u_{j_k} \longrightarrow \bar{u}$ as $k \to +\infty$.

Define the sequence $\{v_j\}_j \subseteq K$ as follows:

$$v_j = \begin{cases} u_{j_k}, & \text{if } j = j_k \text{ (for some } k \in \mathbb{N}) \\ \bar{u}, & \text{otherwise} \end{cases}$$

By construction, $v_j \longrightarrow \bar{u}$ as $j \longrightarrow +\infty$. Moreover, since $F_j \xrightarrow{\Gamma} F$

(*) $$F(\bar{u}) \overset{(i)}{\leq} \liminf_{j \to +\infty} F_j(v_j) \leq \liminf_{k \to +\infty} F_{j_k}(u_{j_k}) = \liminf_{j \to +\infty} F_j(u_j) = \liminf_{j \to +\infty} \inf_X F_j$$

and, for any $u \in X$, there exists a sequence $\{\bar{u}_j\}_j$ s.t.

$$F(u) \overset{(ii)}{\geq} \limsup_{j \to +\infty} F_j(\bar{u}_j) \overset{\text{def}}{\geq} \limsup_{j \to +\infty} \inf_X F_j .$$

Since it holds for any fixed $u \in X$, then

(**) $$\inf_X F \geq \limsup_{j \to +\infty} \inf_X F_j .$$

By (*) and (**), $F(\bar{u}) = \inf_X F$ and the thesis is achieved by the previous steps.

$\square$

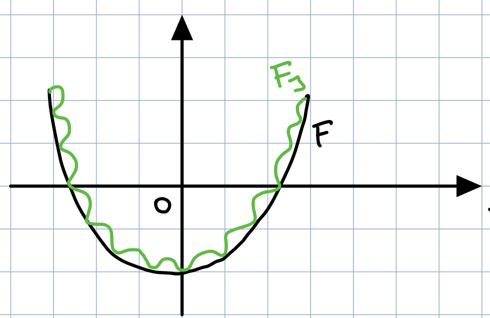
Remarks:

- Note that the Γ-limit F has minimum since it inherits coercivity from the equi-coercivity of $\{F_j\}_j$.
- In the previous proof we constructed by hand the minimizer $\overline{u}$.
- The previous theorem guarantees the convergence of global minima. Note that, in general, Γ-convergence does not guarantee the convergence of local minima.

Example: Let $X = \mathbb{R}$ and let $\begin{aligned} F_j : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 + \sin(jx) \end{aligned}$, $j \in \mathbb{N}$.

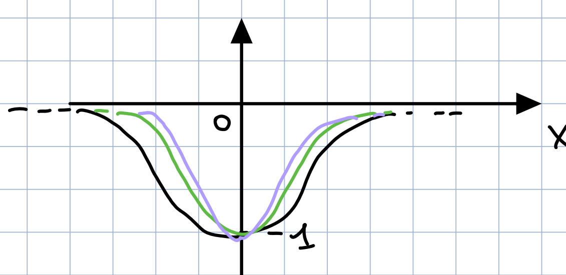
One can prove that $\sin(jx) \xrightarrow{\Gamma} -1$ and, since x^2 is continuous in $\mathbb{R}$, then $F_j \xrightarrow{\Gamma} x^2 - 1$, by previous theorems.

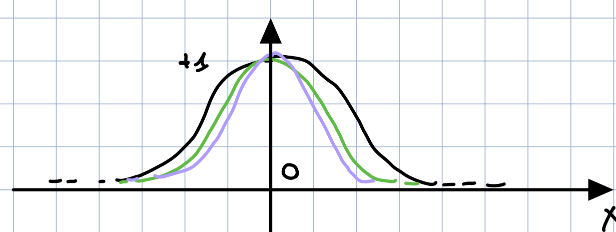
It is then clear the presence of issues at a local level, being the parabola $F(x) = x^2 - 1$ approximated by an oscillating sequence



This example shows also that the equi-coercivity is necessary (everything works fine in $[0, 2\pi]$)

Example: Let $\begin{aligned} F_j : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto -e^{-jx^2} \end{aligned}$ and $\begin{aligned} G_j : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto +e^{-jx^2} \end{aligned}$, $j \in \mathbb{N}$.





Ex: Show that $F_j \xrightarrow{\Gamma} \begin{cases} 0 & , \text{ if } x \in \mathbb{R} \setminus \{0\} \\ -1 & , \text{ if } x = 0 \end{cases}$, while $G_j \xrightarrow{\Gamma} 0$.

What can we say about the pointwise and uniform convergences?

- The previous example shows that the Γ-limit of a sequence of continuous functionals can be only lower semicontinuous (it is not possible e.g. for the uniform convergence).
- Moreover, the Γ-limit of the sequence of antisymmetric functions differs from the Γ-limit of the starting sequence.

<u>Prop</u>: The Γ-limit is unique (if it exists).

<u>Proof</u>: Let $F_j, F, G : X \longrightarrow \overline{\mathbb{R}}$ satisfy

i) $F_j \xrightarrow{\Gamma} F$ ii) $F_j \xrightarrow{\Gamma} G$.

Fix $u \in X$ and $\{u_j\}_j$ a recovery sequence for u w.r.t. F. Then

(o) $F(u) \overset{(ii)_F}{=} \lim_{j \to +\infty} F_j(u_j) \geq \liminf_{j \to +\infty} F_j(u_j) \overset{(i)_G}{\geq} G(u)$.

Analogously, let $\{v_j\}_j$ be a recovery sequence for u w.r.t. G. Then

(oo) $G(u) \overset{(ii)_G}{=} \lim_{j \to +\infty} F_j(v_j) \overset{(i)_F}{\geq} F(u)$.

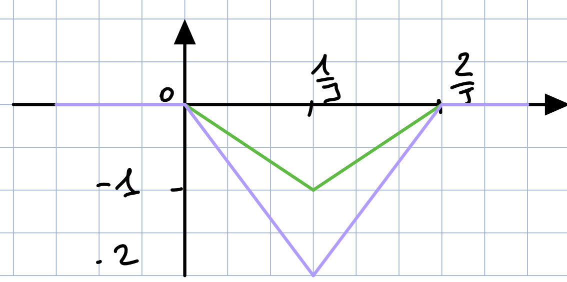
By (o) and (oo), $F(u) = G(u)$ and the result follows by the arbitrariness of u. $\square$

<u>Example</u>: Let $F_j = \begin{cases} F_{1j} & , \text{ if } j \text{ is even} \\ F_{2j} & , \text{ if } j \text{ is odd} \end{cases}$, where

$F_{1j}(x) = \begin{cases} 0 , & \text{if } x \notin (0, \frac{2}{j}) \\ -1 , & \text{if } x = \frac{1}{j} \\ \text{linear and continuous}, & \text{elsewhere} \end{cases}$, $F_{2j}(x) = \begin{cases} 0 , & \text{if } x \notin (0, \frac{2}{j}) \\ -2 , & \text{if } x = \frac{1}{j} \\ \text{linear and continuous}, & \text{elsewhere} \end{cases}$

Then, $\{F_j\}_j$ cannot Γ-converge, otherwise there will be two limits

$F_1 = \begin{cases} 0 , & \text{if } x \neq 0 \\ -1 , & \text{if } x = 0 \end{cases} \neq F_2 = \begin{cases} 0 , & \text{if } x \neq 0 \\ -2 , & \text{if } x = 0 \end{cases}$



5) A local characterization of the Γ-limit

After defining a criterion for Γ-convergence ((i)+(ii)), we want to provide a local characterization of the Γ-limits, w.r.t. the terms of any sequence $\{F_j\}_j$.

Def: Let (X,d) be a metric space, let $F_j : X \longrightarrow \overline{\mathbb{R}}$ and fix $u \in X$. We define

- $\left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(u) \doteq \inf\left\{ \liminf_{j\to+\infty} F_j(u_j) : u_j \longrightarrow u \text{ in } X \right\}$
- $\left(\Gamma\text{-}\limsup_{j\to+\infty} F_j\right)(u) \doteq \inf\left\{ \limsup_{j\to+\infty} F_j(u_j) : u_j \longrightarrow u \text{ in } X \right\}$

· Note that both $\Gamma\text{-}\liminf_{j\to+\infty} F_j$ and $\Gamma\text{-}\limsup_{j\to+\infty} F_j$ always exist and satisfy

$$\boxed{\Gamma\text{-}\liminf_{j\to+\infty} F_j \le \Gamma\text{-}\limsup_{j\to+\infty} F_j}.$$

Th: Let (X,d) be a metric space and let $F_j, F : X \longrightarrow \overline{\mathbb{R}}$, $j \in \mathbb{N}$. Then

$$F_j \xrightarrow{\Gamma} F \iff \Gamma\text{-}\liminf_{j\to+\infty} F_j = \Gamma\text{-}\limsup_{j\to+\infty} F_j = F$$

Proof: ($\Rightarrow$) Assume that $F_j \xrightarrow{\Gamma} F$. Fix $u \in X$ and any $u_j \longrightarrow u$ in X. Then

$F(u) \overset{(i)}{\le} \liminf_{j\to+\infty} F_j(u_j)$, and since it holds for any converging sequence, then

(*) $$F(u) \le \left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(u)$$

· Let now $\{\overline{u}_j\}_j$ be a recovery sequence for F satisfying $\overline{u}_j \longrightarrow u$. Then,

(**) $$\left(\Gamma\text{-}\limsup_{j\to+\infty} F_j\right)(u) \overset{def}{\le} \limsup_{j\to+\infty} F_j(\overline{u}_j) \overset{(ii)}{\le} F(u).$$

By (*) and (**) we get the thesis.

($\Leftarrow$) Assume that $\Gamma\text{-}\liminf_{j\to+\infty} F_j = \Gamma\text{-}\limsup_{j\to+\infty} F_j = F$ and fix $u \in X$. Then, for any $\varepsilon > 0$ there exists a sequence $\{\overline{u}_j\}_j$ such that

$\limsup_{j\to+\infty} F_j(\overline{u}_j) \le \left(\Gamma\text{-}\limsup_{j\to+\infty} F_j\right)(u) + \varepsilon = F(u) + \varepsilon$ and by the arbitrariness of ε (ii) holds.

Then, the thesis is achieved since

$$F(u) = \left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(u) \overset{def}{\le} \liminf_{j\to+\infty} F_j(u_j) \quad \forall\, u_j \longrightarrow u \text{ in } X. \qquad \square$$