Γ-convergence: a 50 years long story (1975-2025)

05.11.2024

References:
- a) An Introduction to Γ-convergence - Dal Maso
- b) Γ-convergence for Beginners - Braides

1) The Direct Methods in the Calculus of Variations

In the next lines we assume, for simplicity, to be in a metric setting. In fact, once we work in metric spaces, the topology can be described through sequences.

· Let (X, d) be a metric space and consider a given functional $F : X \longrightarrow \overline{\mathbb{R}}$. We aim to solve the following minimum problem:

$$\boxed{\min \{ F(u) : u \in X \}}$$

<u>Def</u>: A sequence $\{u_j\}_j \subseteq X$ is called a minimizing sequence for F if

$$\lim_{j \to +\infty} F(u_j) = \inf \{ F(u) : u \in X \} < \infty .$$

first request (=)   second request (<)

<u>Remarks</u>:
1) In general the sequence $\{u_j\}_j$ may not converge to a limit $u \in X$
2) We work with the infimum because it always exists (while, in general, the minimum not)
3) Since X is a metric space, there exists always a sequence $\{u_j\}_j \subseteq X$ satisfying

$$\lim_{j \to +\infty} F(u_j) = \inf \{ F(u) : u \in X \},$$

but, in general, this limit can be not a finite number

<u>Q</u>: Under which conditions a minimizing sequence $\{u_j\}_j \subseteq X$ converges to a limit $u \in X$?

A: We can focus on two different cases:

CASE 1 We can work on compact sets. In fact, in a metric setting, the compactness of the set X (or, of a subset K containing the sequence $\{u_j\}_j$) would be equivalent to the condition of sequential compactness: any sequence in X has a convergent subsequence.

Q: Let $\{u_j\}_j \subseteq X$, $\bar{u} \in X$ and $\{u_{j_k}\}_k \subseteq \{u_j\}_j$ and assume that $u_j \longrightarrow \bar{u}$ in X (for simplicity, $\{u_{j_k}\}_k$ is relabeled $\{u_j\}_j$).

- Under which conditions, if $\{u_j\}_j$ is a minimizing sequence for F,

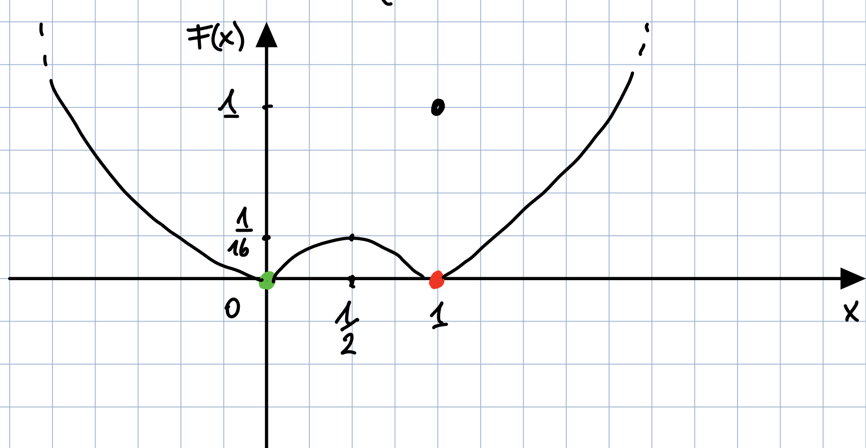
$$\underset{F\left(\lim_{j\to+\infty} u_j\right)}{\underset{\parallel}{F(\bar{u})}} \overset{?}{=} \underset{\lim_{j\to+\infty} F(u_j)}{\underset{\parallel}{\inf\{F(u) : u \in X\}}} \quad ?$$

A: Note that $\inf\{F(u) : u \in X\} \leq F(\bar{u})$ is always satisfied, but the condition $F(\bar{u}) \leq \inf\{F(u) : u \in X\}$ may fail.

Example: Assume $X = \mathbb{R}$, endowed with the Euclidean metric, and consider

$$F : \mathbb{R} \longrightarrow \mathbb{R} \qquad \text{and} \qquad \{x_j\}_j \doteq \left\{1 + \frac{1}{j}\right\}_j$$

$$x \longmapsto \begin{cases} x^2(x-1)^2, & x \neq 1 \\ 1, & x = 1 \end{cases}$$



Ex: Show that $\{x_j\}_j$ is a minimizing sequence for F, but it does not converge to the minimizer of F.

Ex: Consider
$$F: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^2 e^{-x}$$
and study its critical points.

Show that $\{j\}_j$ is a minimizing sequence that does not converge in $\mathbb{R}$.

Def: Let $\{u_j\}_j \subseteq X$ converge (up to subsequences) to $\overline{u} \in X$ and let $F: X \longrightarrow \mathbb{R}$. We say that F is (sequentially) lower semicontinuous if
$$F(\overline{u}) \leq \liminf_{j \to +\infty} F(u_j).$$

The property of lower semicontinuity can be generalized to the setting of topological spaces as follows:

Def: Let (X, τ_X) be a topological space and let $F: X \longrightarrow \overline{\mathbb{R}}$.
We say that F is lower semicontinuous on X if $\forall x \in X$
$$F(x) \leq \liminf_{y \to x} F(y) = \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} F(y)$$

set of all neighbourhoods of x

(or equivalently)
$$\text{if } \forall t < F(x) \; \exists \, U \in \mathcal{U}(x) \text{ s.t. } \forall y \in U \;\; F(y) > t.$$

· By definition it follows immediately that the topological lower semicontinuity implies the sequential one.

The converse is true if X satisfies the first axiom of countability:
Every neighbourhood of any $x \in X$ contains a neighbourhood from a countable family: $\forall x \in X \;\; \exists \{N_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}(x)$ s.t. $\forall N \in \mathcal{U}(x) \; \exists \, i \in \mathbb{N}$ s.t. $N_i \subseteq N$.

Remark: Every metric space is first-countable.

Let us provide a counterexample of a sequentially lower semicontinuous functional, that is not (topologically) lower semicontinuous.

Example: Let Ω be a bounded, open subset of $\mathbb{R}^n$, with $n \geq 1$ and define

- $C^\infty(\Omega) \doteq \{ u : \Omega \longrightarrow \mathbb{R} : \text{smooth - with continuous derivatives of any order} \}$
- $C_c^\infty(\Omega) = \{ u \in C^\infty(\Omega) : \overline{\text{supp}(u)} \text{ is compact} \}$, where $\text{supp}(u) = \{ x \in \Omega : u(x) \neq 0 \}$
- $L^2(\Omega) = \{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable s.t. } \int_\Omega u^2(x)\,dx < \infty \}$, which is a Banach space endowed with the norm

$$\|u\|_{L^2} \doteq \left(\int_\Omega u^2(x)\,dx \right)^{\frac{1}{2}}, \quad u \in L^2(\Omega)$$

- $H^1(\Omega) = W^{1,2}(\Omega) \doteq \{ u \in L^2(\Omega) : Du \in L^2(\Omega) \}$, where Du is the "weak" derivative of u, meaning that

$$\int_\Omega Du(x) \cdot v(x)\,dx = - \int_\Omega u(x) \cdot v'(x)\,dx \quad \forall v \in C_c^1(\Omega).$$

$H^1(\Omega)$ is still an infinite dimensional Banach space, endowed with the norm

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}, \quad u \in H^1(\Omega)$$

- $H_0^1(\Omega) = W_0^{1,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1}}$.

To provide a counterexample we must work in a setting in which the first axiom of countability is not satisfied. We then consider the space $H_0^1(\Omega)$ with the weak topology $\tau_{weak} \doteq \sigma(H_0^1, H^{-1})$.

We remind that the weak topology on $H_0^1(\Omega)$ is the "cheapest" topology on $H_0^1(\Omega)$ for which any collection of maps $\{F_i\}_{i \in I}$, with $F_i \in H^{-1}(\Omega) \doteq \{ G : H_0^1(\Omega) \longrightarrow \mathbb{R} \text{ linear and bounded} \}$, is continuous.

Prop: $\forall \bar{u} \in H^1_0(\Omega)$, $\forall \varepsilon > 0$ $\forall \{F_1, \dots, F_k\} \subseteq H^{-1}(\Omega)$, the set
$$V = \left\{ u \in H^1_0(\Omega) : \left| F_i(u - \bar{u}) \right| < \varepsilon \ \ \forall i \in \{1, \dots, k\} \right\}$$ is a neighbourhood of $\bar{u}$ for the weak topology τ_{weak}.

- Let $K = \left\{ u \in H^1_0(\Omega) : \|u\|_{L^2(\Omega)} \geq 1 \right\}$ and let $\mathbb{1}_K(u) \doteq \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{if } x \in H^1_0(\Omega) \setminus K \end{cases}$

be the "indicator function". To conclude we show that $\mathbb{1}_K$ is sequentially lower semicontinuous but not (topologically) lower semi continuous, by means of the following result:

Prop: $\mathbb{1}_K$ is (seq.) lower semicontinuous if and only if K is (seq.) closed.

a) We first show that K is sequentially closed: let $\{u_j\}_j \subseteq (K, \tau_{weak})$ (we now consider the subspace topology) such that $u_j \longrightarrow u$ weakly in $H^1_0(\Omega)$.
We show that $u \in K$ by proving that $\|u\|_{L^2(\Omega)} \geq 1$. It is a consequence of the following well-known result

Th (Rellich-Kondrachov): The space $H^1(\Omega)$ compactly embeds into $L^2(\Omega)$, meaning that any bounded sequence in $H^1(\Omega)$ strongly converges in $L^2(\Omega)$ (up to a subsequence).

Then, $u_j \longrightarrow u$ strongly in $L^2(\Omega)$ and so $\|u\|_{L^2(\Omega)} = \lim_{j \to +\infty} \|u_j\|_{L^2(\Omega)} \geq 1$

b) We finally show that K is not closed w.r.t. τ_{weak}.
Fix $u_0 = 0 \in H^1_0(\Omega) \setminus K$ and consider a neighbourhood of u_0 (see the proposition above)
$$V = \left\{ u \in H^1_0(\Omega) : | F_i(u) | < \varepsilon \ \forall i = 1, \dots, k \right\}$$
for any $\varepsilon > 0$ and $F_1, \dots, F_k \in H^{-1}(\Omega)$.
To conclude we show that $V \cap K \neq \emptyset$, meaning that $u_0 = 0 \in \overline{K}^{\tau_{weak}}$ (the closure of K w.r.t. the weak topology).
Since $H^1_0(\Omega)$ is an infinite dimensional Banach space, then $\bigcap_{i=1}^{k} \mathrm{Ker}(F_i) \neq \{0\}$.

Otherwise $\exists\ \phi : H^1_0(\Omega) \longrightarrow \mathbb{R}^k$ injective

$u \longmapsto (F_1(u), \dots, F_k(u))$

and so $H^1_0(\Omega)$ would be finite dimensional.

Then, there exists $\overline{u} \in \bigcap_{i\in 1}^{k} \ker(F_i)$ s.t. $\overline{u} \neq 0 = u_0$.

Notice also that there exists $t >> 1$ such that

$$\| t\overline{u} \|_{L^2(\Omega)} = t \|\overline{u}\|_{L^2(\Omega)} \geq 1.$$

Therefore, $t\overline{u} \in V \cap K$ and $V \cap K \neq \{0\}$.

__Remark__: If $\{u_j\}_j$ is a minimizing sequence, then $\lim_{j\to+\infty} F(u_j)$ exists and so the lower semicontinuity of F is equivalent to

$$F(\overline{u}) \leq \liminf_{j\to+\infty} F(u_j) = \lim_{j\to+\infty} F(u_j) = \inf\{F(u) : u \in X\}.$$

__Th__ (Weierstrass): Let (X, τ_X) be a (seq.) topologically compact space and let $F : X \longrightarrow \overline{\mathbb{R}}$ be (seq.) lower semicontinuous.

Then, there exist $\min\{F(u) : u \in X\}$.

CASE 2 Instead of asking a stronger condition on X (we will see that in many cases it will not be possible), we can find an equivalent compactness condition for the functional F

__Def__: $F : X \longrightarrow \overline{\mathbb{R}}$ is (topologically) coercive if, for any $t \in \mathbb{R}$, the sublevel

$$\{F < t\} = \{u \in X : F(u) < t\}$$

is relatively compact (or precompact).

To understand why the coercivity of F is related with the compactness of X, we remind that (in metric spaces) it is equivalent to the following condition.

Def: F is sequentially coercive if, any sequence $\{u_j\}_j \subseteq X$ satisfying $\sup_{j\in\mathbb{N}} F(u_j) < +\infty$, admits a converging subsequence in X.

Remark: If we work in normed spaces $(X, \|\cdot\|)$ the coercivity of F is equivalent to
$$\lim_{\|u\|\to+\infty} F(u) = +\infty$$
or, equivalently, $\exists\ c \in \mathbb{R}^+$ s.t. $F(u) \geq \|u\| + c \quad \forall u \in X$.

Th (Tonelli): Let (X, τ_X) be a topolog. space and let $F : X \longrightarrow \overline{\mathbb{R}}$ be (seq.) coercive and (seq.) lower semicontinuous. Then, there exists
$$\min\{F(u) : u \in X\}.$$

Proof: Let $\{u_j\}_j$ be a minimizing sequence for F satisfying
$$\inf_X F \leq F(u_j) \leq \inf_X F + \frac{1}{j} \qquad \forall j \in \mathbb{N}.$$
Then, denoted $t \doteq \inf_X F + 1$, $\{u_j\}_j \subseteq \{F < t\}$ → relatively compact

and so there exists a subsequence $\{u_{j_k}\}_k \subseteq \{u_j\}_j$ convergent to $\overline{u} \in X$.

Moreover,
$$\inf_X F \underset{\text{properties of inf}}{\leq} F(\overline{u}) \overset{\text{lower semic.}}{\leq} \liminf_{j\to+\infty} F(u_j) \underset{\{u_j\}_j \text{ min. sequence}}{=} \lim_{j\to+\infty} F(u_j) = \inf_X F \qquad \square$$

Def: We define Direct Method in the calculus of variations the procedure:

Coercivity + lower semicontinuity $\Rightarrow$ existence of minimizers.

Remarks:

1) The properties of compactness (or coercivity) and lower semicontinuity are opposite requirements. For instance, if we consider the sequential coercivity of a given functional $F: X \longrightarrow \overline{\mathbb{R}}$ (X topological space), then it is easier to be verified if we have many converging sequences, while the sequential lower semicontinuity of F is more easily satisfied if we have few converging sequences. We should then find an appropriate topology for X that balances these aspects (it will be part of the problem in what follows).

2) Note that neither the Weierstrass theorem nor the Tonelli theorem guarantee the uniqueness of the minimizer. Moreover, $\min_X F$ could be "$+\infty$".

- If we move to the setting of topological vector spaces, we find a sufficient condition for the uniqueness of the minimizer and finiteness of the minimum.

Def: We define $(X, \tau_X, +, \cdot)$ a topological vector space (T.V.S. in short) if

a) (X, τ_X) is a topological space;

b) $(X, +, \cdot)$ is a vector space over $\mathbb{R}$ (or any topological field $\mathbb{K}$);

c) The vector space operations $+ : X \times X \longrightarrow X$ and $\cdot : \mathbb{R} \times X \longrightarrow X$ are continuous.

Examples: Every normed space is a topological vector space, considering the topology induced by the distance induced by the norm (so also every Banach and Hilbert space).

- Also spaces whose topology is not induced by a norm can be T.V.S. For instance: $C^\infty(\Omega)$, $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$, $\mathcal{D}'(\Omega)$ ...

Def: Let X be a T.V.S. We say that $F: X \longrightarrow \overline{\mathbb{R}}$ is (strictly) convex if $\exists\, \bar{x} \in X$ s.t. $F(\bar{x}) < +\infty$ and $\boxed{F(tx + (1-t)y) \underset{(<)}{\leq} t F(x) + (1-t) F(y)}$ for every $t \in (0,1)$ and for every $x, y \in X$ s.t. $F(x), F(y) < +\infty$.

Prop: Let X be a T.V.S. and let $F : X \longrightarrow \overline{\mathbb{R}}$ be strictly convex. Then, F has at most one minimizer in X.

Proof: By contradiction, assume the existence of two distinct (global) minimizers $u, v \in X$ s.t. $u \neq v$. Then,

$$F(u) = F(v) = \min_X F < +\infty.$$

If we consider the point $\frac{u}{2} + \frac{v}{2} \in X$, then, by strict convexity,

$$F\left(\frac{u}{2} + \frac{v}{2}\right) < \frac{1}{2}\left[F(u) + F(v)\right] = \min_X F$$

and this yields a contradiction. Therefore, $u = v$. $\square$

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2) The problem of the Γ-convergence

SETTING: Metric spaces (for simplicity)

We want to study the behaviour of a family (sequence) of minimum problems depending on a real parameter $\varepsilon > 0$ or, equivalently, on a discrete parameter $j \in \mathbb{N}$ (if we consider the real parameter, the limit case will be $\varepsilon \longrightarrow 0$ while, in the discrete setting, $j \longrightarrow +\infty$).

Note that the continuous (ε) and discrete settings (j) are related by the following rule: to any family of real parameters $\varepsilon \in \mathbb{R}^+$, we associate a sequence $\{\varepsilon_j\}_j$ s.t.

$$\varepsilon_j \longrightarrow 0 \quad \text{as} \quad j \longrightarrow +\infty.$$

- Let X_j be a metric space, let $F_j : X_j \longrightarrow \overline{\mathbb{R}}$ be a sequence of functionals and consider the family of problems

$$\inf\left\{F_j(u) : u \in X_j\right\}, \quad j \in \mathbb{N}.$$

AIM: As j increases, we would like these problems to be approximated by a limit theory, described by a problem of the form

$$\min\left\{F(u) : u \in X\right\}.$$

The limit space X should be a metric space large enough to contain any space X_j
$$X_j \subseteq X \quad \forall j \in \mathbb{N}.$$
In this way, we don't have to face the problem of defining the convergence of functionals belonging to different spaces.
Without loss of generality, we will then consider $X_j = X \quad \forall j \in \mathbb{N}$, by identifying any functional $F_j : X_j \longrightarrow \overline{\mathbb{R}}$ with
$$\tilde{F}_j : X \longrightarrow \overline{\mathbb{R}}$$
$$u \longmapsto \begin{cases} F_j(u) & \text{if } u \in X_j \\ +\infty & \text{if } u \in X \setminus X_j \end{cases}$$
<u>Remark</u>: $\inf\limits_{X_j} F_j = \inf\limits_{X} F.$

· Let then (X, d) be a metric space and $F_j : X \longrightarrow \overline{\mathbb{R}}$ and assume the existence of a sequence of minimizers $\{u_j\}_j \subseteq X$ for $\{F_j\}_j$, that is
$$F_j(u_j) = \inf\{F_j(u) : u \in X\} \quad \forall j \in \mathbb{N}.$$
$$\Longrightarrow \qquad \lim_{j \to +\infty} \left[F_j(u_j) - \inf_X F_j \right] = 0. \qquad (*)$$

(Note that, in general, such sequence may not exist).
· As in the case of one single problem, we need a compactness condition that ensures the convergence (up to subsequences) of $\{u_j\}_j$.
<u>Def</u>: We say that the functionals $\{F_j\}_j$ are equi-coercive if
$$\forall t \in \mathbb{R} \quad \exists\, K \subseteq X \text{ compact s.t. } \{F_j \le t\} \subseteq K \quad \forall j \in \mathbb{N}$$
(The compact K depends only on t but not on j).
· If $\{F_j\}_j$ are equi-coercive, then there exists a compact set K s.t. $\{u_j\}_j \subseteq K$ and, by compactness (in this metric setting it is equivalent to the sequential compactness), there exists $\overline{u} \in X$ s.t (up to subsequences)
$$u_j \longrightarrow \overline{u} \quad \text{in } X \quad (\text{w.r.t. the metric } d).$$

We then ask under which hypotheses there exists a limit functional $F : X \longrightarrow \mathbb{R}$ s.t. $\boxed{F(\bar{u}) = \min_X F}$.

STEP 1 (Lower bound)

First, we want to obtain a lower bound for the limit behaviour of the liminf of the sequence $\{F_j(u_j)\}_j$, of the form

$$F(\bar{u}) \le \liminf_{j\to+\infty} F_j(u_j) \overset{(*)}{=} \liminf_{j\to+\infty} \left(\inf_X F_j\right).$$

Def: We define **Γ-liminf inequality** the following condition:

$$\boxed{\forall\, u \in X \;\; \forall \{v_j\}_j \subseteq X \text{ s.t. } v_j \longrightarrow u \text{ in } X \Rightarrow F(u) \le \liminf_{j\to+\infty} F_j(v_j)}.$$

STEP 2 (Upper bound)

Next, we want to obtain an upper bound for limsup $\{F_j(u_j)\}_j$, of the form

$$\limsup_{j\to+\infty} F_j(u_j) \le \inf_X F \le F(\bar{u}).$$

↳ always true

By $(*)$, this is equivalent to

$$\limsup_{j\to+\infty} \left(\inf_X F_j\right) \le F(u) \quad \underline{\forall\, u \in X}.$$

Note that this requirement is global $(\forall\, u \in X)$, which is convenient to <u>localize</u> in a neighbourhood of any point $u \in X$ as follows: $\forall\, u \in X \;\; \forall\, \delta > 0$

$$\limsup_{j\to+\infty} \left(\inf \{ F_j(v) : d(u,v) < \delta \} \right) \le F(u).$$

This last condition is stronger w.r.t. the previous one and can be rephrase in terms of sequences as follows.

Def: We define **Γ-limsup inequality** the following condition:

$$\boxed{\forall\, u \in X \;\; \exists \{\bar{v}_j\}_j \subseteq X \text{ s.t. } \bar{v}_j \longrightarrow u \text{ in } X \text{ s.t. } F(u) \ge \limsup_{j\to+\infty} F_j(\bar{v}_j)}.$$

CONCLUSIONS: Given the sequence of functionals $\{F_j\}_j$, with $F_j : X \longrightarrow \overline{\mathbb{R}}$ $\forall j \in \mathbb{N}$, if we can find a functional $F : X \longrightarrow \overline{\mathbb{R}}$ and a converging sequence of minimizers $\{u_j\}_j$, $u_j \longrightarrow \overline{u}$, such that the liminf inequality and the limsup inequality are satisfied, then

$$F(\overline{u}) \overset{\text{liminf ineq.}}{\leq} \liminf_{j \to +\infty} F_j(u_j) \overset{(*)}{=} \liminf_{j \to +\infty} \inf_X F_j \leq \limsup_{j \to +\infty} \inf_X F_j \overset{\text{limsup ineq.}}{\leq} \inf_X F \leq F(\overline{u})$$

$\longrightarrow$ always true

that is, $F(\overline{u}) = \inf_X F$ (i.e. $\overline{u}$ is a minimizer of F).

Then, we deduce at the same time that:

1) (Existence) the limit problem $\min\{F(u) : u \in X\}$ has a solution $\overline{u} \in X$

2) (Convergence of the minimum values)

$$\inf_X F_j \longrightarrow \min_X F \quad \text{as } j \longrightarrow +\infty$$

3) (Convergence of the minimizers) up to subsequences, the sequence of minimizers for $\{F_j\}_j$ converges to a minimizer of F in X, i.e.

$$u_j \longrightarrow \overline{u} \quad \text{in } X \quad \text{as } j \longrightarrow +\infty \quad \text{(up to subsequences).}$$

Def: Let (X, d) be a metric space and let $F_j, F : X \longrightarrow \overline{\mathbb{R}}$, $j \in \mathbb{N}$.

We say that the sequence $\{F_j\}_j$ Γ-converges to F ($F_j \xrightarrow{\Gamma} F$ in short) with respect to the metric d if:

i) (Γ-liminf inequality) $\forall u \in X$ $\forall$ $u_j \longrightarrow u$ in $X \Rightarrow F(u) \leq \liminf_{j \to +\infty} F_j(u_j)$

ii) (Γ-limsup inequality) $\forall u \in X$ $\exists$ $v_j \longrightarrow u$ in X s.t. $F(u) \geq \limsup_{j \to +\infty} F_j(v_j)$.

Remark: Conditions i) + ii) are equivalent to i) + ii'), where

ii') (Γ-lim equality) $\forall u \in X$ $\exists$ $v_j \longrightarrow u$ in X s.t. $F(u) = \lim_{j \to +\infty} F_j(v_j)$.

The sequence $\{v_j\}_j$ is called "recovery sequence". Moreover, it holds that

Γ-convergence + equi-coerciveness $\Rightarrow$ convergence of minimum problems.