Second Part – References: a, b, c, e

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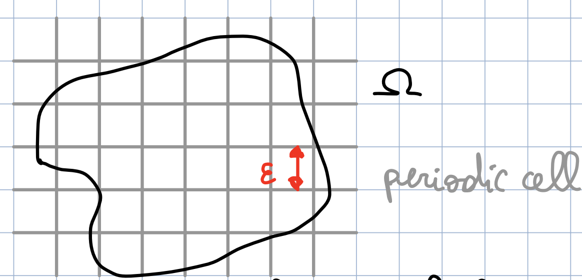
1) Homogenization of multiple integrals

The object of homogenization theory is the description of the macroscopic properties of media with fine microstructures, like composites, fiber materials, stratified or porous media, finitely damaged materials, materials with cracks or holes.

· From a pure mathematical point of view, we are going to study the asymptotic behaviour of fast-oscillating integral functionals of the form

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx \qquad \text{as } \varepsilon \longrightarrow 0$$

with $\Omega \subseteq \mathbb{R}^n$ open and bounded, $\varepsilon \in \mathbb{R}^+$ (scale factor) and $f : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty)$ periodic in the first variable, i.e. $f(x + e_i, \xi) = f(x, \xi)$ $\forall i = 1, \dots, n$ $\forall \xi \in \mathbb{R}^n$ a.e. $x \in \Omega$.



If the scale ε becomes very small, the behaviour of the material may not be interesting and the properties of the medium can be described by replacing F_ε by a homogenized integral $F_{hom}(u) = \int_\Omega f_{hom}(\nabla u(x))\, dx$.

In real applications, this can happen when we consider a cellular non-linearly hyperelastic material. Here Ω represents the reference configuration of the body, ε the side-length of the periodic cell and F_ε the elastic energy of the material subject to a displacement u.

We are going to see that the previous approximation will make sense once we will provide a sort of convergence between the minimizers and minima of the problems

$$\min\{F_\varepsilon(u) : u \in X\} \quad \text{and} \quad \min\{F_{hom}(u) : u \in X\}.$$

The first step in this theory consists in finding a candidate for the space X.

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be open and bounded and let $1 \leq p \leq +\infty$.

The functional setting we are interested in must involve derivatives.

__Def__: We define the Sobolev spaces

- $W^{1,p}(\Omega) \doteq \left\{ u \in L^p(\Omega) : \|u\|_{W^{1,p}} < +\infty \right\}$, with $\|u\|_{W^{1,p}} \doteq \|u\|_{L^p} + \|\nabla u\|_{L^p}$.

  $\nabla u = (D_1 u, \dots, D_n u)$ is called the "weak gradient" (or "distributional gradient").
- $W_0^{1,p}(\Omega) \doteq \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}}}$.

__Remark__: $W^{1,p}_{(0)}$ are Banach spaces (reflexive if $p \neq 1, \infty$ and separable if $p \neq \infty$) and Hilbert iff $p=2$.

An important result concerning the space $W_0^{1,p}$ is the following one.

__Th__: (Poincaré inequality) If Ω is open and bounded, then there exists $c = c(p, \Omega) > 0$ s.t.

$$\|u\|_{L^p} \leq c \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega).$$

__Cor__: An equivalent norm in $W_0^{1,p}(\Omega)$ is $\|\nabla \cdot\|_{L^p}$.

The case $p = \infty$ is quite delicate, because of the absence of reflexivity. We then need:

__Def__: We say that: 1) $u_j \overset{*}{\rightharpoonup} u$ in L^∞-weak $*$ if

$$\int_\Omega u_j v \, dx \longrightarrow \int_\Omega u v \, dx \qquad \forall v \in L^1(\Omega)$$

2) $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}$-weak $*$ if

a) $u_j \overset{*}{\rightharpoonup} u$ in L^∞-weak $*$

b) $\nabla u_j \overset{*}{\rightharpoonup} \nabla u$ in $(L^\infty\text{-weak } *)^n$.

any component of the weak gradient converges to the corresponding component.

We also remind the following crucial result.

__Th__: (Banach-Alaoglu-Bourbaki)

Original statement: For any topological vector space $(X, \tau, +, \cdot)$ with continuous dual space $X' \doteq \{ F : X \longrightarrow \mathbb{R} \text{ linear and continuous} \}$ the polar of any neighbourhood U of the origin in X, defined as

$$U^\circ = \left\{ F \in X' : \sup_{u \in U} |F(u)| \leq 1 \right\},$$

is compact in the weak$*$-topology of X' $\sigma(X', X)$.

· **Normed spaces**: If $(X, \|\cdot\|_X)$ is a normed spaces the closed unit ball in the continuous dual space X' is compact w.r.t. the weak-$*$ topology.

Cor: Let $\{u_j\}_j \subseteq W^{1,\infty}(\Omega)$ be bounded, i.e. $\exists$ $K \in \mathbb{R}^+$ independent of j s.t.

$$\|u_j\|_{W^{1,\infty}} \leq K \qquad \forall j \in \mathbb{N}$$

(N.B. if $p = +\infty$, $\|u\|_{W^{1,\infty}} \doteq \|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} = \operatorname{ess\,sup}_\Omega |u| + \operatorname{ess\,sup}_\Omega |\nabla u|$).

Then, by the B-A-B theorem there exist $u \in L^\infty(\Omega)$ and $\phi \in (L^\infty(\Omega))^n$ s.t.

$$(\text{up to subsequences}) \begin{cases} u_j \overset{*}{\rightharpoonup} u & \text{in } L^\infty\text{-weak } * \\ \nabla u_j \overset{*}{\rightharpoonup} \phi & \text{in } (L^\infty\text{-weak } *)^n. \end{cases}$$

Moreover, working with appropriated test functions, one can show that $\phi = \nabla u$.

Remark: In what follows $\Omega \subseteq \mathbb{R}^n$ is open and bounded and $1 \leq p < +\infty$.

· We now need to properly define the class of integral functionals

$$F(u) = \int_\Omega f(x, \nabla u(x))\,dx.$$

· For the functional F to be well-defined, we can require that the integrand

$$f : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty] \text{ is Borel-measurable}$$

and set

$$F : W^{1,p}(\Omega) \longrightarrow [0, +\infty]$$
$$u \longmapsto \int_\Omega f(x, \nabla u(x))\,dx$$

being composition of B-meas. funct.

· Remark: By hypotheses, the map $x \longmapsto f(x, \nabla u(x))$ is Borel-measurable and non negative.

Then, $f(\cdot, \nabla u(\cdot)) \in L^1(\Omega)$ and so F is well-defined.

· We aim to find natural and non-restrictive assumptions for f that guarantee the existence of minima for F (by means of the Direct Methods in the calculus of variations).

1. We need to control the growth of f and we will assume the existence of $c \in \mathbb{R}^+$:

$$f(x, \xi) \geq c|\xi|^p \qquad \text{a.e. } x \in \Omega \ \forall \xi \in \mathbb{R}^n.$$

This first assumption is quite natural and guarantees the coercivity of F.

In fact, if $\{u_j\}_j \subseteq W^{1,p}(\Omega)$ satisfies $\sup_j F(u_j) < +\infty$, then

$$(\infty >)\ F(u_j) \doteq \int_\Omega f(x, \nabla u_j(x))\,dx \overset{①}{\geq} \int_\Omega c|\nabla u_j(x)|^p dx = c\|\nabla u_j\|^p_{L^p} \quad \forall j \in \mathbb{N}$$

and so $\sup_j \|\nabla u_j\|_{L^p} < +\infty$.

We will see soon that the natural space of definition of F will the subset of $W^{1,p}$ with zero boundary $(W^{1,p}_0)$. In that setting, by the Poincaré inequality

$$\sup_j \|\nabla u_j\|_{L^p} < +\infty \Rightarrow \sup_j \|u_j\|_{W^{1,p}} < \infty$$

and, if $p \neq 1$ (we need reflexivity at this point), once we know that $\{u_j\}_j$ is bounded in $W^{1,p}$, then exists a subsequence weakly convergent in the space.

N.B. The previous argument suggests that the natural topology is the weak one of $W^{1,p}$.

To be able to apply the Direct Methods we also need lower semicontinuity in this topology. Consider again any $p \in [1, +\infty)$. In the next result we show that if the integrand f is lower semicontinuous (in the II° variable), then F is l. sc. in the strong topology and, if f is also convex (in the II° variable), then the lower semicontinuity holds in the weak topology too.

Th: Let $1 \leq p < +\infty$ and assume that $f : \Omega \times \mathbb{R}^n \longrightarrow [0, +\infty]$ satisfies:

i) f is Borel-measurable;

ii) for a.e. $x \in \Omega$ the map $\begin{aligned} \mathbb{R}^n &\longrightarrow [0,+\infty] \\ \xi &\longmapsto f(x,\xi) \end{aligned}$ is lower semicontinuous.

Then, $\begin{aligned} F : W^{1,p}(\Omega) &\longrightarrow [0,+\infty] \\ u &\longmapsto \int_\Omega f(x, \nabla u(x))\,dx \end{aligned}$ is lower semicontinuous in $W^{1,p}$-strong.

If, in addition

iii) for a.e. $x \in \Omega$ the map $\begin{aligned} \mathbb{R}^n &\longrightarrow [0,+\infty] \\ \xi &\longmapsto f(x,\xi) \end{aligned}$ is convex, then

F is lower semicontinuous in $W^{1,p}$-weak.

<u>Proof</u>: Let $u_j, u \in W^{1,p}(\Omega)$ satisfy $u_j \longrightarrow u$ (strongly). We want to show that

$$F(u) \leq \liminf_{j \to +\infty} F(u_j).$$

Note that, if $\liminf_{j \to +\infty} F(u_j) = +\infty$, then the conclusion is trivial. We assume that $\liminf_{j\to+\infty} F(u_j) < +\infty \Rightarrow \exists \{u_{j_k}\}_k \subseteq \{u_j\}_j$ s.t. $\liminf_{j\to+\infty} F(u_j) = \lim_{k\to+\infty} F(u_{j_k})$ and, in particular, that $\boxed{\liminf_{j\to+\infty} F(u_j) = \liminf_{k\to+\infty} F(u_{j_k})}$ (*)

· Since $\{u_j\}_j$ strongly converges to u in $W^{1,p}(\Omega)$, then (by construction):

1) $\{u_{j_k}\}_k$ still strongly converges to u in $W^{1,p}(\Omega)$;
2) $\{\nabla u_{j_k}\}_k$ strongly converges to ∇u in $L^p(\Omega)$ (by definition);
3) $\{\nabla u_{j_k}(x)\}_k$ converges a.e. to $\nabla u(x)$ in Ω (up to a further subsequence).

Then, by hypothesis (ii) and Fatou's lemma

$$F(u) \doteq \int_\Omega f(x, \nabla u(x))\, dx \overset{(ii)}{\leq} \int_\Omega \liminf_{k\to+\infty} f(x, \nabla u_{j_k}(x))\, dx \overset{\text{Fatou}}{\leq} \liminf_{k\to+\infty} \int_\Omega f(x, \nabla u_{j_k}(x))\, dx$$

$$= \liminf_{k\to+\infty} F(u_{j_k}) \overset{(*)}{=} \liminf_{j\to+\infty} F(u_j).$$

· The second part of the proof follows from the following result of functional analysis.

<u>Th</u>: Let $(X, \|\cdot\|_X)$ be a Banach space and let $F: X \longrightarrow \overline{\mathbb{R}}$ be convex. Then,

F is lower semicontinuous in the strong topology of X if and only if

F is lower semicontinuous in the weak topology of X.

<u>Proof</u>: F is lower semicontinuous in the strong topology of X if and only if the sublevels $\{F \leq t\}$ are closed in the strong topology of X $(\forall t)$, while

F is lower semicontinuous in the weak topology of X if and only if the sublevels $\{F \leq t\}$ are closed in the weak topology of X $(\forall t)$.

· We remind that in any Banach space any convex set is strongly closed if and only if it is weakly closed.

Since F is convex by hypothesis, then any set $\{F \leq t\}$ is convex □