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Remark: The properties stated above in the framework of metric spaces, such as the "Fundamental Theorem of Γ-convergence", hold in any topological space.

Def: Let (X, τ_X) be a topological space, let $F_j : X \longrightarrow \overline{\mathbb{R}}$, $j \in \mathbb{N}$, and for any $u \in X$ denote $\mathcal{N}(u)$ the set of all open neighbourhoods of u.

We define:

- $$\left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(u) \doteq \sup_{U\in\mathcal{N}(u)} \liminf_{j\to+\infty} \inf_{v\in U} F_j(v)$$
- $$\left(\Gamma\text{-}\limsup_{j\to+\infty} F_j\right)(u) \doteq \sup_{U\in\mathcal{N}(u)} \limsup_{j\to+\infty} \inf_{v\in U} F_j(v).$$

If there exists $F : X \longrightarrow \overline{\mathbb{R}}$ such that

$$\left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(u) = \left(\Gamma\text{-}\limsup_{j\to+\infty} F_j\right)(u) = F(u) \quad \forall\, u \in X,$$

we say that $\{F_j\}_j$ Γ-converges to $F = \Gamma\text{-}\lim\limits_{j\to+\infty} F_j$ in X, w.r.t. the topology τ_X.

EX: Show that if $(X, \tau_X) = (X, d)$ is a metric space, then the topological definition is equivalent to the sequential one.

· For applications it will not be easy to work with the previous definition, which involves neighbourhoods. However, in few cases it is useful to have the definition of Γ-limit in terms of the topology of X, as in the following proposition.

Prop: $\Gamma\text{-}\liminf\limits_{j\to+\infty} F_j$, $\Gamma\text{-}\limsup\limits_{j\to+\infty} : X \longrightarrow \overline{\mathbb{R}}$ are lower semicontinuous.

Proof: (We only study the case of $\Gamma\text{-}\liminf\limits_{j\to+\infty}$. The other case is analogous). Fix $u \in X$, then

$$\liminf_{v\to u}\left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(v) \doteq \sup_{U\in\mathcal{N}(u)} \inf_{v\in U}\left(\sup_{V\in\mathcal{N}(v)} \liminf_{j\to+\infty} \inf_{w\in V} F_j(w)\right)$$

w.l.o.g $V \subseteq U$

$$\left(\inf_U \le \inf_{V\subseteq U}\right)$$

$$\geq \sup_{U\in\mathcal{N}(u)} \cancel{\inf_{v\in U}} \cancel{\sup_{U\supseteq V\in\mathcal{N}(v)}} \liminf_{j\to+\infty} \inf_{w\in U} F_j(w)$$

$$= \sup_{U\in\mathcal{N}(u)} \liminf_{j\to+\infty} \inf_{w\in U} F_j(w) \doteq \left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(u) \qquad \square$$

Cor: Let (X, τ_X) be a topological space and let $F_j, F : X \longrightarrow \overline{\mathbb{R}}$ be such that $F = \Gamma\text{-}\lim\limits_{j\to+\infty} F_j$. Then, F is lower semicontinuous.

6) Relaxation - lower semicontinuous envelope

In the previous sections we showed that

$$F_j = F \ (\forall j \in \mathbb{N}) \xrightarrow{\Gamma} F \iff F \text{ is lower semicontinuous.}$$

However, the sequence $\{F\}_j$ has always a Γ-limit, and we now want to represent it.

<u>Def</u>: Let (X, τ_X) be a topological space and let $F : X \longrightarrow \overline{\mathbb{R}}$.

We define the relaxation of F (or lower semicontinuous envelope) at $u \in X$ the functional

$$\overline{F}(u) \doteq \sup \{ G(u) : G \text{ is lower semicontinuous and } G \leq F \}.$$

<u>Remark</u>: $\overline{F}$ is lower semicontinuous (the supremum preserves the lower semicontinuity).

Moreover, $\overline{F} \leq F$ and $\overline{F} \geq G$ for any G lower semicontinuous s.t. $G \leq F$.

· Note that the definition of $\overline{F}$ involves the behaviour of F in the whole space X.

By means of the topology τ_X we can however provide a local characterization of $\overline{F}$.

<u>Prop</u>: Let (X, τ_X) be a topological space and let $F : X \longrightarrow \overline{\mathbb{R}}$. Then

$$\overline{F}(u) = \sup_{U \in \mathcal{N}(u)} \inf_{v \in U} F(v).$$

<u>EX</u>: Prove the previous proposition.

· By the previous proposition, it is clear the following result.

<u>Prop</u>: Let (X, τ_X) be a topological space and let $F : X \longrightarrow \overline{\mathbb{R}}$. Then

$$F_j = F \ (\forall j \in \mathbb{N}) \xrightarrow{\Gamma} \overline{F}.$$

<u>Proof</u>: Since F_j does not depend on j $(F_j = F \ \forall j \in \mathbb{N})$, then $\forall u \in X$

$$\left(\Gamma\text{-}\liminf_{j \to +\infty} F_j\right)(u) = \sup_{U \in \mathcal{N}(u)} \liminf_{j \to +\infty} \inf_{v \in U} \underbrace{F_j(v)}_{= F(v)} = \underbrace{\sup_{U \in \mathcal{N}(u)} \inf_{v \in U} F(v)}_{= \overline{F}(u)} = \left(\Gamma\text{-}\limsup_{j \to +\infty} F_j\right)(u)$$

$\square$

· Once we move back to the stronger setting of metric spaces, the previous prosition provides another characterization of the relaxation of F in terms of sequences.

Prop: Let (X,d) be a metric space and let $F: X \to \overline{\mathbb{R}}$. Then

$$\overline{F}(u) = \min\left\{ \liminf_{j\to+\infty} F(u_j) : u_j \to u \text{ in } X \right\} \quad \forall u \in X.$$

Proof: By the previous proposition, taking $F_j = F \ \forall j \in \mathbb{N}$, then $\forall u \in X$

$$\overline{F}(u) = \left(\Gamma\text{-}\liminf_{j\to+\infty} F_j\right)(u) \overset{\text{Metric space}}{=} \inf\left\{ \liminf_{j\to+\infty} F_j(u_j) : u_j \to u \text{ in } X \right\}$$

$$\overset{F_j=F}{=} \inf\left\{ \liminf_{j\to+\infty} F(u_j) : u_j \to u \text{ in } X \right\}.$$

Moreover, by Γ-convergence, the infimum is attained (at least for any recovery sequence $\{\overline{u}_j\}_j$ satisfying (ii') or (ii)), and then the thesis easily follows.

$\square$

· The following result compares Γ-limits of sequences interrelated.

Prop: Let (X,τ_X) be a topological space and let $F_j, G_j : X \to \overline{\mathbb{R}}$ satisfy $F_j \leq G_j \quad \forall j \in \mathbb{N}$. Then

$$\boxed{\Gamma\text{-}\liminf_{j\to+\infty} F_j \leq \Gamma\text{-}\liminf_{j\to+\infty} G_j} \quad \text{and} \quad \boxed{\Gamma\text{-}\limsup_{j\to+\infty} F_j \leq \Gamma\text{-}\limsup_{j\to+\infty} G_j}.$$

In particular, if $\{F_j\}_j$ and $\{G_j\}_j$ Γ-converge to F and G (resp.), then $F \leq G$.

Moreover, if $H: X \to \overline{\mathbb{R}}$ is lower semicontinuous and $H \leq G_j \quad \forall j \in \mathbb{N}$, then

$$H \leq \Gamma\text{-}\liminf_{j\to+\infty} G_j \leq \Gamma\text{-}\limsup_{j\to+\infty} G_j$$

and so, if $\{G_j\}_j$ Γ-converges to G, then $H \leq G$.

· Note that the previous proposition can be improved if we compare the sequences $\{\overline{F_j}\}_j$ and $\{F_j\}_j$, keeping in mind that $\overline{F_j} \leq F_j \quad \forall j \in \mathbb{N}$ by definition.

In this particular case we obtain this strong result, useful in the applications.

Prop: Let (X,τ_X) be a topological space and let $F_j : X \to \overline{\mathbb{R}} \quad \forall j \in \mathbb{N}$. Then

$$\Gamma\text{-}\liminf_{j\to+\infty} F_j = \Gamma\text{-}\liminf_{j\to+\infty} \overline{F_j} \quad \text{and} \quad \Gamma\text{-}\limsup_{j\to+\infty} F_j = \Gamma\text{-}\limsup_{j\to+\infty} \overline{F_j}.$$

In particular,

$$\boxed{\{F_j\}_j \ \Gamma\text{-converges to } F \text{ if and only if } \{\overline{F_j}\}_j \ \Gamma\text{-converges to } F.}$$

Ex: Prove the two propositions above.

We conclude this section with the following application of the direct methods in calc. var.

Th: Let (X, τ_X) be a topological space and let $F: X \longrightarrow \overline{\mathbb{R}}$ be mildly coercive (i.e. there exists $K \subseteq X$ compact, $K \neq \emptyset$ s.t. $\inf_X F = \inf_K F$). Then

1) there exists $\min\{\overline{F}(u) : u \in X\}$

2) $\min_X \overline{F} = \inf_X F$

3) $\overline{u} \in \{u \in X : \overline{F}(u) = \min_X \overline{F}\}$ if and only if there exists a minimizing sequence for F $\{u_j\} \subseteq X$ (i.e. $\lim_{j \to +\infty} F(u_j) = \inf_X F$) such that $u_j \longrightarrow \overline{u}$ in X.

Proof: Let $F_j = F$ for any $j \in \mathbb{N}$. Then, by previous results,

$$F_j \xrightarrow{\Gamma} \overline{F}, \text{ which is lower semicontinuous.}$$

Then, by the Fundamental Theorem of Γ-convergence, we get (1) and (2) and also the implication "$\Leftarrow$" of (3). We conclude showing the reverse implication in (3).

· Fix $\overline{u} \in \{u \in X : \overline{F}(u) = \min_X \overline{F}\}$. Then, by Γ-convergence, there exists a recovery sequence $\{u_j\}_j \subseteq X$ for $\overline{F}$ such that $u_j \longrightarrow \overline{u}$ in X and

$$\lim_{j \to +\infty} F(u_j) \overset{F_j = F}{=} \lim_{j \to +\infty} F_j(u_j) \overset{(ii')}{=} \overline{F}(\overline{u}) = \min_X \overline{F} \overset{(2)}{=} \inf_X F.$$

The conclusion follows by the arbitrariness of $\overline{u}$. $\square$

7) Γ-convergence by subsequences

The last section concerning the theoretical part is devoted to two important result: the "Urysohn property of Γ-convergence" and a compactness theorem.

Before stating and proving these results, we notice as follows.

Remark: Let (X, τ_X) be a topological space and let $F_j : X \longrightarrow \overline{\mathbb{R}}$, $j \in \mathbb{N}$.

If $\{F_{j_k}\}_k$ is a subsequence of $\{F_j\}_j$, then

$$\Gamma\text{-}\liminf_{j \to +\infty} F_j \leq \Gamma\text{-}\liminf_{k \to +\infty} F_{j_k} \quad \text{and} \quad \Gamma\text{-}\limsup_{k \to +\infty} F_{j_k} \leq \Gamma\text{-}\limsup_{j \to +\infty} F_j.$$

However, if $F = \Gamma\text{-}\lim_{j \to +\infty} F_j$, then $\{F_{j_k}\}_k$ Γ-converges to F $\forall \{j_k\}_k \subseteq \mathbb{N}$ increasing.

Proposition: (Urysohn property of Γ-convergence)

Let (X, τ_X) satisfy the first axiom of countability (e.g. metric spaces), and let $F_j, F : X \longrightarrow \overline{\mathbb{R}}$, $j \in \mathbb{N}$. Then

$\{F_j\}_j$ Γ-converges to F $\iff$ $\forall \{F_{j_k}\}_k \subseteq \{F_j\}_j$ $\exists \{F_{j_{k_n}}\}_n \subseteq \{F_{j_k}\}_k$ s.t.

$$F_{j_{k_n}} \xrightarrow{\Gamma} F \quad \text{as } n \to +\infty.$$

Proof: $(\Rightarrow)$ Notice that for any $\{j_k\}_k \subseteq \mathbb{N}$ increasing

$$F = \Gamma\text{-}\liminf_{j\to+\infty} F_j \overset{\text{Remark}}{\leq} \Gamma\text{-}\liminf_{k\to+\infty} F_{j_k} \leq \Gamma\text{-}\limsup_{k\to+\infty} F_{j_k} \leq \Gamma\text{-}\limsup_{j\to+\infty} F_j = F.$$

$(\Leftarrow)$ Assume, by contradiction, that

$$F_j \overset{\Gamma}{\not\longrightarrow} F, \text{ i.e. } \exists\, u \in X \text{ s.t.}$$

(a) $\left(\Gamma\text{-}\liminf\limits_{j\to+\infty} F_j\right)(u) < F(u)$, or

(b) $\left(\Gamma\text{-}\limsup\limits_{j\to+\infty} F_j\right)(u) > F(u)$.

- In the first case, there exists a sequence $\{u_j\}_j \subseteq X$ s.t. $u_j \longrightarrow u$ in X and

$$\liminf_{j\to+\infty} F_j(u_j) < F(u)$$

$$\Rightarrow \exists \{F_{j_k}\}_k \subseteq \{F_j\}_j \text{ s.t. } \lim_{k\to+\infty} F_{j_k}(u_{j_k}) = \liminf_{j\to+\infty} F_j(u_j).$$

$$\lim_{k\to+\infty} F_{j_k}(u_{j_k}) = \limsup_{k\to+\infty} F_{j_k}(u_{j_k})$$

Then, $\left(\Gamma\text{-}\limsup\limits_{k\to+\infty} F_{j_k}\right)(u) < F(u)$ and we get a contradiction by the Remark.

- In the second case, there exists $U \in \mathcal{N}(u)$ s.t.

$$\limsup_{j\to+\infty} \inf_{v\in U} F_j(v) > F(u).$$

As before, there exists $\{F_{j_k}\}_k \subseteq \{F_j\}_j$ s.t.

$$F(u) < \limsup_{j\to+\infty} \inf_{v\in U} F_j(v) = \lim_{k\to+\infty} \inf_{v\in U} F_{j_k}(v) = \liminf_{k\to+\infty} \inf_{v\in U} F_{j_k}(v)$$

and, passing to the supremum, we get a contradiction by the Remark, since

$$\Gamma\text{-}\liminf_{k\to+\infty} F_{j_k}(u) > F(u).$$

$\square$

Theorem: Let (X, τ_X) satisfy the second axiom of countability (i.e. there exists a countable basis for τ_X). Then, every sequence $\{F_j\}_j$, $F_j : X \to \overline{\mathbb{R}}$ has a Γ-convergent subsequence.

Proof: By hypotheses $\exists$ $\mathcal{B} = \{U_m\}_m$ a countable basis (of open sets) of τ_X.

Fix $m \in \mathbb{N}$ and consider the sequence

$$\left\{ \inf_{v \in U_m} F_j(v) \right\}_j \subseteq \overline{\mathbb{R}}.$$

Since $\overline{\mathbb{R}}$ is compact, then there exists $\{F_{j_k}\}_k \subseteq \{F_j\}_j$ such that

$$(*) \qquad \lim_{k \to +\infty} \inf_{v \in U_m} F_{j_k}(v) \text{ exists (in } \overline{\mathbb{R}}).$$

$(*)$ holds $\forall m \in \mathbb{N}$ and, by a diagonal argument, we construct

$$\{F_{j_{k_\ell}}\}_\ell \subseteq \{F_{j_k}\}_k \text{ s.t. } \lim_{\ell \to +\infty} \inf_{v \in U} F_{j_{k_\ell}}(v) \text{ exists } \forall U \in \mathcal{B}.$$

- Fix now $u \in X$ and denote $\mathcal{B}(u) \doteq \{U \in \mathcal{B} \text{ s.t. } u \in U\}$ and

$$F(u) = \sup_{U \in \mathcal{B}(u)} \lim_{\ell \to +\infty} \inf_{v \in U} F_{j_{k_\ell}}(v) \doteq \left(\Gamma\text{-}\lim_{\ell \to +\infty} F_{j_{k_\ell}}\right)(u).$$

By the arbitrariness of u, we get the thesis. $\square$

Remark: The two previous results can be used in combination as follows:

$\forall \{F_j\}_j$ we extract a subsequence $\{F_{j_k}\}_k \subseteq \{F_j\}_j$, which Γ-converges by the previous theorem. If we then show that the Γ-limit is the same for any subsequence, then we inherit Γ-converge for $\{F_j\}_j$, by the Uryshon property.

- Since in applications the functionals F_j, $j \in \mathbb{N}$, will often depend on a continuous parameter $\varepsilon \in \mathbb{R}^+$, we will need the following definition.

Def: We say that $\{F_\varepsilon\}_\varepsilon$ Γ-converges to F if $\forall \{\varepsilon_j\}_j \subseteq \mathbb{R}^+$ s.t. $\varepsilon_j \to 0$, as $j \to +\infty$, then

$$\boxed{F = \Gamma\text{-}\lim_{j \to +\infty} F_{\varepsilon_j}.}$$