Localised Multilinear Restriction Estimates

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A fundamental result in modern fourier analysis is the celebrated Bennett–Carbery–Tao multilinear restriction estimate: given an ensemble $(S_1, \dots, S_n)$ of compact transversal hypersurfaces in $\mathbb{R}^n$, let E_j denote the fourier extension operator associated with S_j for each $1 \leqslant j \leqslant n$. Then, for all $\varepsilon, R > 0$,

$$\Big\| \prod_{j=1}^{n} E_j f_j \Big\|_{L^{\frac{2}{n-1}}(B_R)} \lesssim_{\varepsilon} R^{\varepsilon} \prod_{j=1}^{n} \|f_j\|_{L^2}.$$

This inequality has found use in a number of important contexts, examples including the linear restriction problem, decoupling theory, and the theory of maximal functions. Following work of Bejenaru, certain localised variants of multilinear restriction estimates, which are sensitive to the geometry of the support of the functions f_j, have been shown to greatly extend the utility of this framework. In this talk, I will describe the localised multilinear restriction estimate and discuss the main aspects of its proof, in particular how the underlying geometry is captured by a certain generalised notion of a Brascamp–Lieb inequality— these being a broad class of estimates on positive multilinear forms, classical examples of which include Hölder's inequality, Young's convolution inequality, and the Loomis–Whitney inequality. I will then illustrate how these localised multilinear restriction estimates may be applied in conjunction with other tools from harmonic analysis in order to establish the complete range (up to endpoints) of off-diagonal estimates for the helical maximal function in $\mathbb{R}^3$, which we define as, given a curve $\gamma : [-1, 1] \to \mathbb{R}^3$,

$$M_\gamma f(x) := \sup_{t>0} \left| \int_{\mathbb{R}} f(x - t\gamma(s))\chi(s) ds \right|,$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a bump function supported on $[-1, 1]$, and we assume that γ has non-vanishing torsion. This is joint work with David Beltran and Jonathan Hickman.