Is complexity an invariant of the representation base?

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Abstract

Measuring the complexity of real numbers is of major importance in computer science. Consider a non-computable real number x, i.e., a real number which cannot be stored on a computer. One can store only an approximation of x, for instance by considering a finite length bitstring representing a prefix w of the binary expansion ω of x. For a fixed approximation error $\varepsilon > 0$, the required length of this bitstring, as a function of ε, depends on the *algorithmic complexity* of the prefix w of the binary expansion achieving error ε. The *algorithmic complexity* of a binary sequence w, often referred to as *Kolmogorov complexity*, is the length of the smallest binary sequence w', for which there exists an algorithm, such that when presented with w' as input, delivers w as output. The algorithmic complexity of the binary expansion of real numbers has been widely studied, but the algorithmic complexity of expansions in bases other than 2 remains poorly understood. Several papers have established an equivalence between the algorithmic complexity of the expansions in different bases $B \in \mathbb{N}$. Here, we study the algorithmic complexity of expansions in noninteger bases, which display a much more sophisticated behavior. This type of expansions, often referred to as β-expansions, have been studied widely in the literature in the context of dynamical systems. We find a relation between the algorithmic complexity of expansions in noninteger bases and the algorithmic complexity of expansions in base 2.

Notation: $A_q := \{0, \ldots, q-1\}$. A_q^*, A_q^ω denote finite and infinite sequences with elements in A_q. $l(w)$ is the length of $w \in A_q^*$. $\omega_{1:n}$ denotes the n-prefix of $\omega \in A_q^\omega$. U is a universal Turing machine. We fix $x \in [0,1]$ and $B \in (1, \infty)$ throughout.

1 Expansions in base B

An expansion of x in base B is a sequence $\omega \in A_{\lceil B \rceil}^\omega$ that satisfies

$$x = \omega_1 B^{-1} + \omega_2 B^{-2} + \ldots + \omega_n B^{-n} + \ldots = \sum_{i=1}^{\infty} \omega_i B^{-i} =: 0.\omega_B \quad (1)$$

Example: $0.75_{10} = 0.11_2 = 0.3_4 = 0.1_{4/3}$.

- $\Sigma_B(x) := \{\omega \in A_{\lceil B \rceil}^\omega : x = 0.\omega_B\}$. If $B < \frac{1+\sqrt{5}}{2}$, $\#\Sigma_B(x) = 2^{\chi_0}$ [EK+90].
- $g_B(x)$ is the lexicographically maximal element of $\Sigma_B(x)$.

2 Algorithmic complexity

- $K[w]$ measures the amount of information in $w \in A_q^*$, according to

$$K[w] := \min \{l(p) : p \in \{0,1\}^*, U(p) = w\}. \quad (2)$$

Let $\omega \in \{0,1\}^\omega$ be generated randomly.

$$K[0000000000000000] \text{ is low,} \quad (3)$$
$$K[3.141592653589793] \text{ is low,} \quad (4)$$
$$K[\omega_{1:10}] \simeq 10, \text{ with high probability.} \quad (5)$$

- For $\omega \in A_q^\omega$, let $K[\omega|n] := K[\omega_{1:n}]$. $\omega \sim \omega' \in A_{q'}^{\omega}$ if

$$K[\omega'|\lceil n \log_{q'}(q) \rceil] = K[\omega|n] + O(1). \quad (6)$$

Example: $\pi \sim 0^\infty \nsim \omega$.

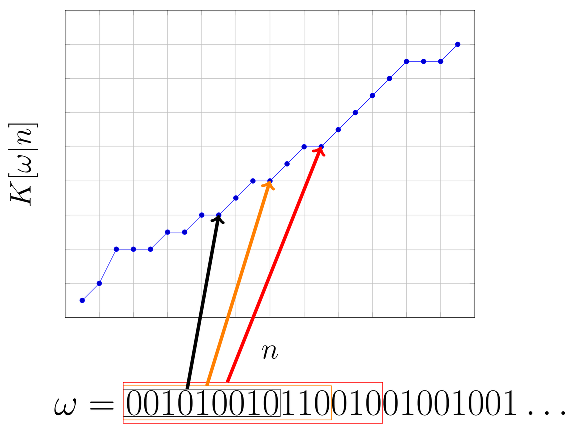


Figure 1: Kolmogorov complexity of initial segments of an infinite string.

3 Relationship between complexity for different $B \in \mathbb{N}$

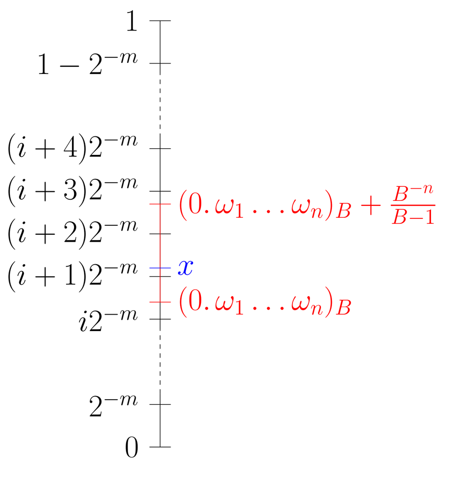
Let $\omega \in A_B^\omega$, $\omega' \in A_2^\omega$, s.t. $0.\omega_B = 0.\omega'_2$. From [Sta02],

$$\omega \sim \omega', \text{ i.e. } K[\omega'|\lceil n \log_2(B) \rceil] = K[\omega|n] + O(1). \quad (7)$$

4 Relationship between complexity for different $B \in \mathbb{Q}$

- For $\omega \in \Sigma_B(x)$, $(0.\omega_{1:n})_B \leq x \leq (0.\omega_{1:n})_B + \frac{B^{-n}}{B-1}$.
- For $w, w' \in \{0,1\}^m$, $w \neq w'$, $|0.w_2 - 0.w'_2| \geq 2^{-m}$ $\Rightarrow$ knowing $\omega_{1:n}$, only $\propto 2^m B^{-n}$ expansions for x in base 2 are possible.
- By setting $m = \lceil \log_2(B) \rceil$, one has

$$K[\omega'|\lceil n \log_2(B) \rceil] \leq K[\omega|n] + O(1). \quad (8)$$



4.1 A dynamical system to generate an expansion with minimal complexity

$\{0.w_B : w \in \{0,1\}^m\}$ is very hard to study $\Rightarrow$ other proof than for (8) needed.

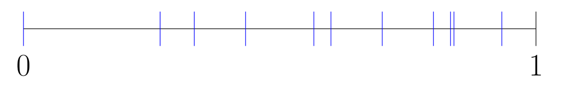


Figure 2: Distribution of values of $(w_1 \ldots w_5)_{1.3}$

- We define constructively an expansion ω of x in base B satisfying

$$K[\omega|n] \leq K[\omega'|\lceil n \log_2(B) \rceil] + O(1), \quad \forall \omega' \ s.t. \ 0.\omega'_2 = x. \quad (9)$$

We generate chunks $c_1, c_2, \ldots$ of ω of length N upon reading of chunks of length $M_1, M_2, \ldots$, with $\Sigma_n := M_1 + \ldots + M_n = \lceil n \log_2(B) \rceil$.

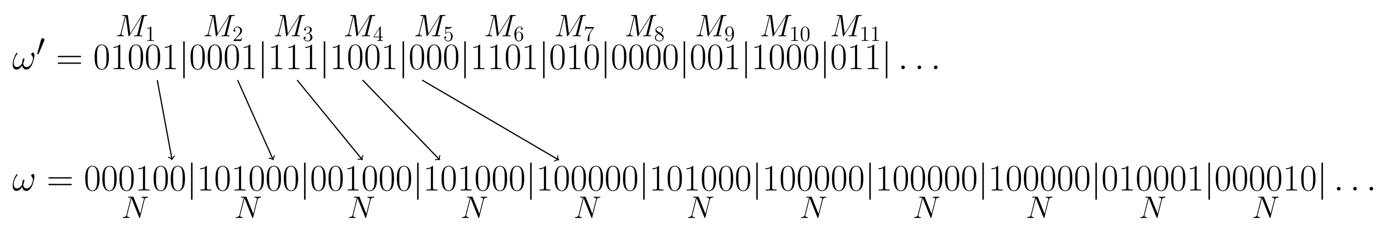


Figure 3: Generation of chunks of β-expansion from chunks of binary expansion.

- This construction of ω relies on two sequences x_n, c_n, with $x_0 = 0$, $c_0 = \epsilon$, and x_n, c_n for $n \geq 1$ generated according to the following procedure.

Algorithm 1 Computation of x_n, c_n from x_{n-1}

1: Take w_n to be the n-th chunk of ω', of length M_n.
2: $y_n \leftarrow x_{n-1} + \frac{B^{Nn}}{2^{\Sigma_n}} \times (0.w_n)_2$.
3: Take c_n to be the first N bits of $g_B(y_n)$.
4: $x_n \leftarrow x_{n-1} - B^{-N}(0.c_n)_B$.

Finally, define $\omega = c_1 c_2 c_3 \ldots$.

4.2 Final relationship

Let $\omega \in A_{\lceil B \rceil}^\omega$ be defined as above, $\omega' \in A_2^\omega$, s.t. $0.\omega_B = 0.\omega'_2$.

$$K[\omega'|\lceil n \log_2(B) \rceil] = K[\omega|n] + O(1). \quad (10)$$

5 Relationship between complexity for different $B \in (1, \infty)$

- (7) is extended to computable $B \in (1, \infty)$,
- extension to all (including noncomputable) $B \in (1, \infty)$:

$$|K[\omega'|\lceil n \log_2(B) \rceil] - K[\omega|n]| \leq K[g_2(\beta)|n] + O(1). \quad (11)$$

6 Applications

- The complexity of sequences generated by the robust A/D conversion algorithm in [Dau+06] can be studied.
- The hierarchy of computational power of RNNs established in [BGS97], based on complexity of the binary expansions of the weights, can be generalized to any representation base.

References

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