EMPIRICAL APPROXIMATION OF THE GAUSSIAN DISTRIBUTION IN $\mathbb{R}^d$

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Abstract. Let $G_1, \dots, G_m$ be independent copies of the standard gaussian random vector in $\mathbb{R}^d$. We show that there is an absolute constant c such that for any $A \subset S^{d-1}$, with probability at least $1 - 2\exp(-c\Delta m)$, for every $t \in \mathbb{R}$,

$$\sup_{x \in A} \left| \frac{1}{m} \sum_{i=1}^{m} 1_{\{\langle G_i, x\rangle \le t\}} - \mathbb{P}(\langle G, x\rangle \le t) \right| \le \Delta + \sigma(t)\sqrt{\Delta}.$$

Here $\sigma(t)$ is the variance of $1_{\{\langle G, x\rangle \le t\}}$ and $\Delta \ge \Delta_0$, where Δ_0 is determined by an unexpected complexity parameter of A that captures the set's geometry (Talagrand's γ_1 functional). The bound, the probability estimate, and the value of Δ_0 are all (almost) optimal.

We use this fact to show that if $\Gamma = \sum_{i=1}^{m} \langle G_i, x\rangle e_i$ is the random matrix that has $G_1, \dots, G_m$ as its rows, then the structure of $\Gamma(A) = \{\Gamma x : x \in A\}$ is far more rigid and well-prescribed than was previously expected.

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