New perspectives on algebra from applied topology

Ezra Miller

Duke University, Department of Mathematics
and Department of Statistical Science

ezra@math.duke.edu

Hypatia Graduate Summer School

CRM, Barcelona

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Outline

1. Data
2. Persistent homology
3. Ordinary persistence: one parameter
4. Multiple parameters: fruit fly wings
5. Multiple parameters: probability distributions
6. Multigraded algebra
7. History of persistent homology

What kinds of data?

Shapes

- 1D: curves (in $\mathbb{R}^2$ or $\mathbb{R}^3$, say)
- 2D: photographs
- 3D: MRI, DTI, SPECT, PET, CAT, integrated photo
  - cricket sclerites
  - brain arteries
  - lung airways
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- (2+1)D: video (.mp4, .mov, ...)
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- neurological
- metabolic
- regulatory (genetic)
- phylogenetic
- physical: road maps, plant roots, neuronal (dendritic), . . .

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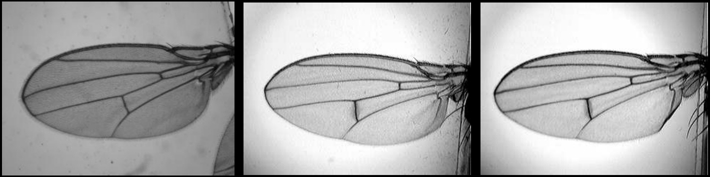
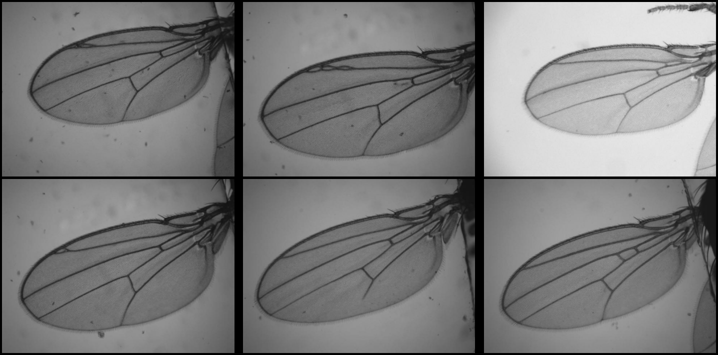
Fruit fly wings

Normal fly wings [images from David Houle's lab]:



Topologically abnormal veins:



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A. apoplanos

courtesy Elen Oneal

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Brain arteries

[Bullitt and Aylward, 2002]

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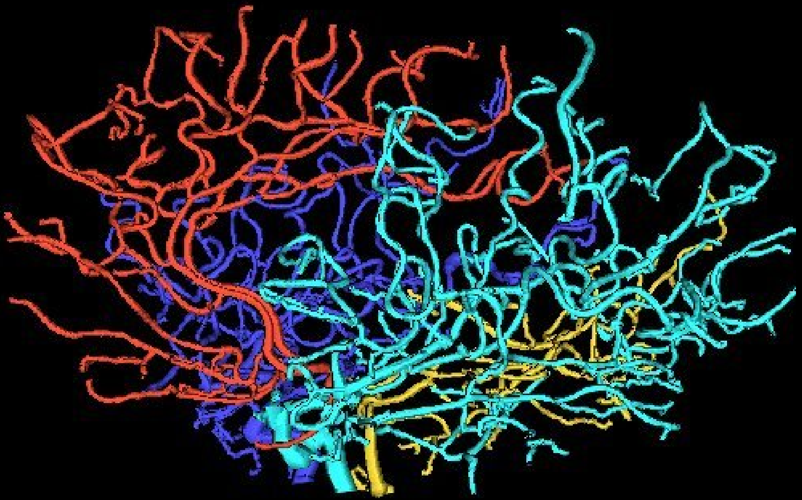

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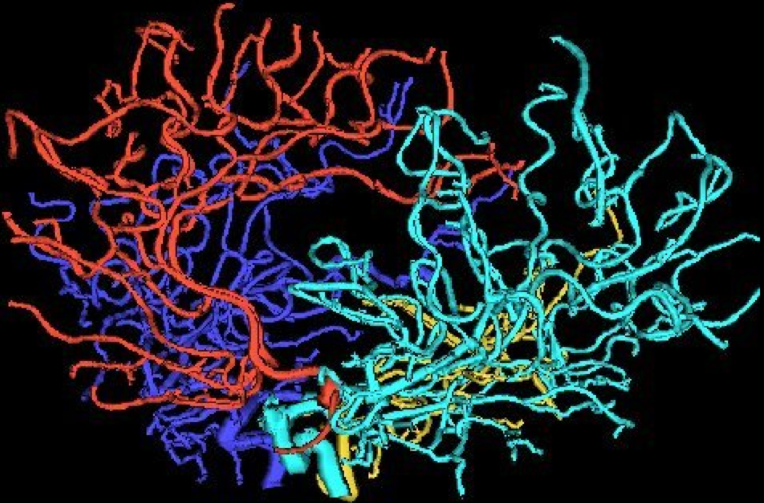

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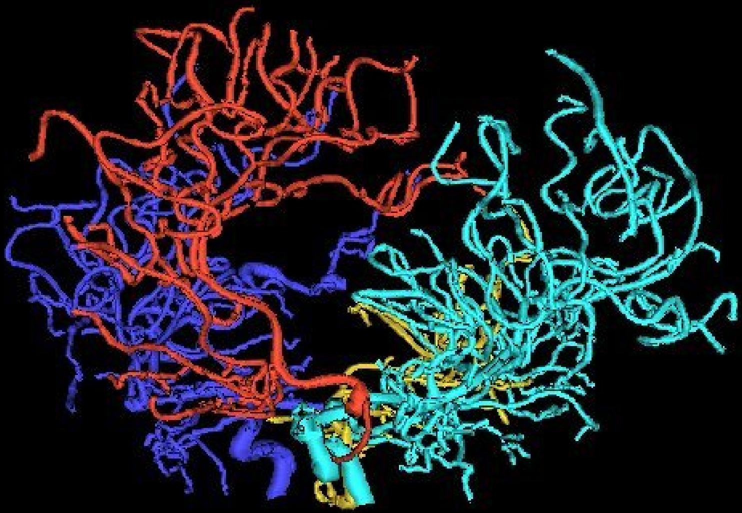

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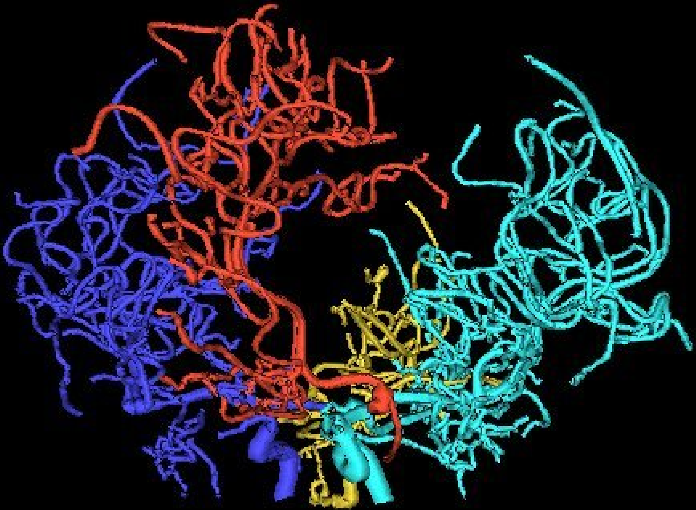

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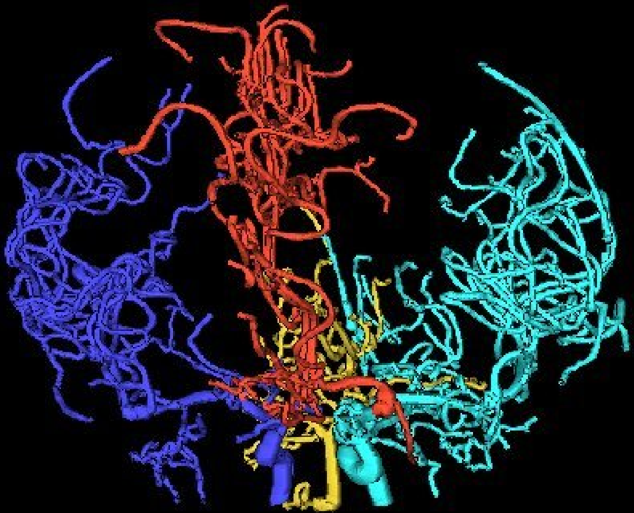

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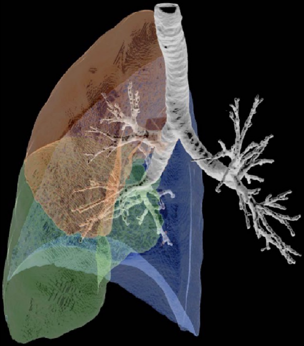
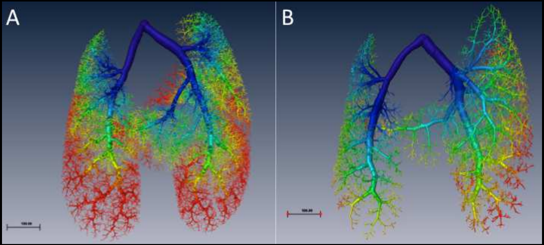
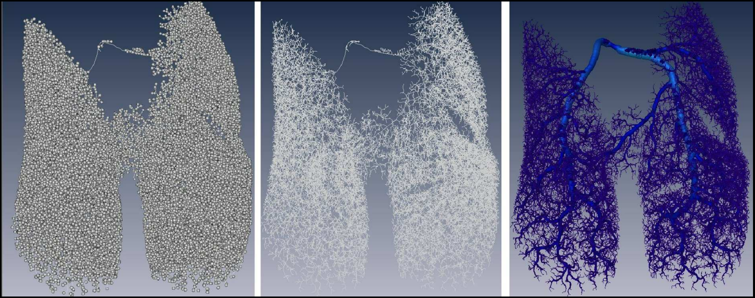
Lung airways (COPD study)



[Belchi, Pirashvili, Conway, Bennett, Djukanovic, Brodzki 2018]

Lung vessels (CDH study)





courtesy Sean McLean

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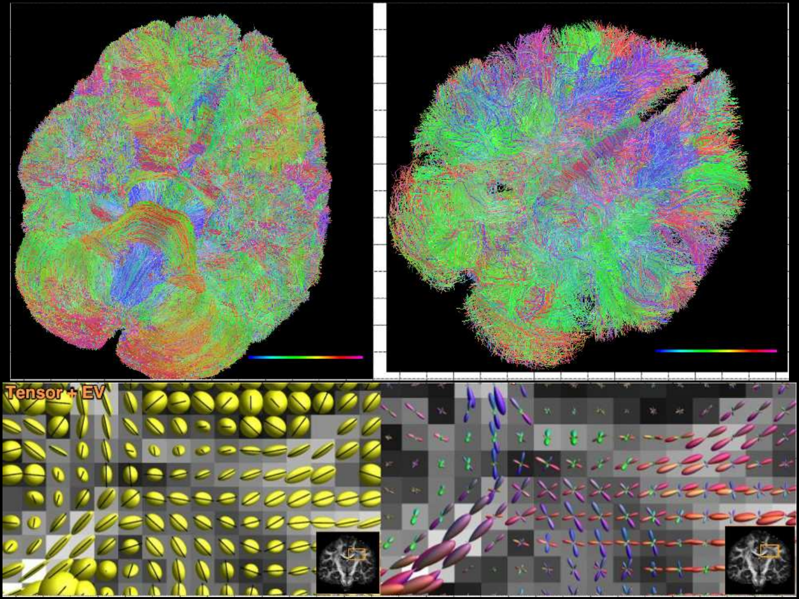
Streamlines from Diffusion Tensor Imaging



courtesy Zhengwu Zhang

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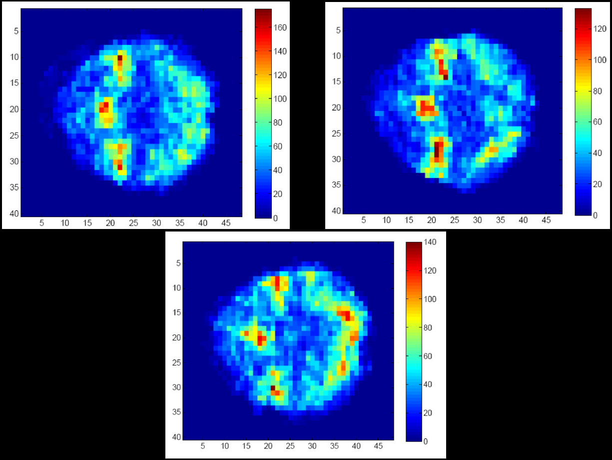
fMRI



courtesy Nicole Lazar

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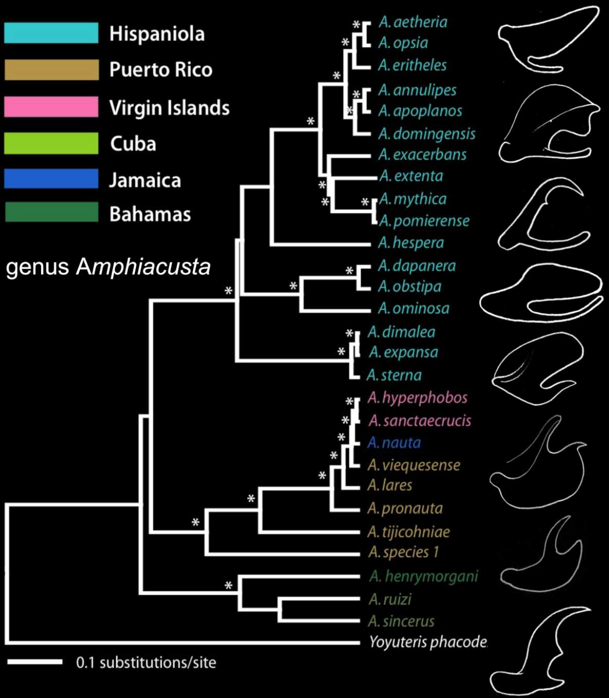

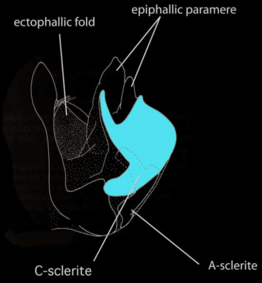

From Oneal, Otte & Knowles, 2010

Drawings by Dan Otte

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Persistent homology

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$.

Def. Q-module over the poset Q:

- family $H = \{H_q\}_{q \in Q}$ of vector spaces over the field $\Bbbk$ with
- homomorphism $H_q \to H_{q'}$ whenever $q \prec q'$ in Q such that
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Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ — 2 discrete or continuous parameters
- probability distributions: $Q = \mathbb{R}^2$ — 2 continuous parameters
- $Q = \mathbb{Z}^n \Leftrightarrow H = \mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-module
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- $Q = \mathbb{R}^n \Leftrightarrow H = \mathbb{R}^n$-graded $\Bbbk[\mathbb{R}^n_+]$-module

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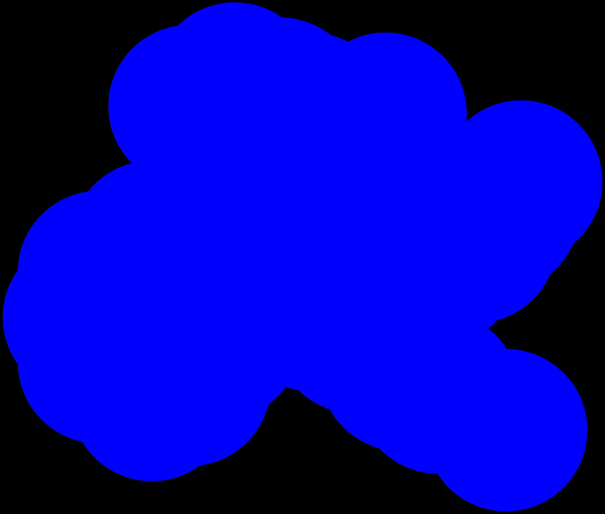
Example: expanding balls



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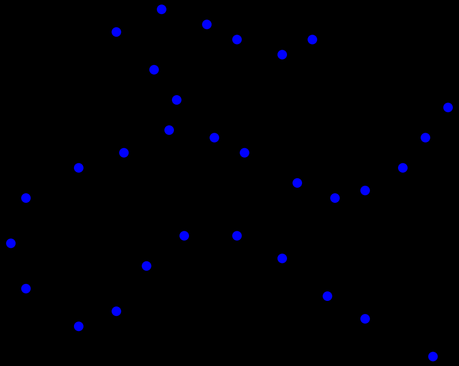

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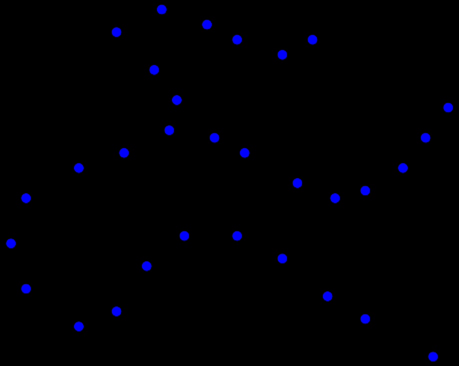
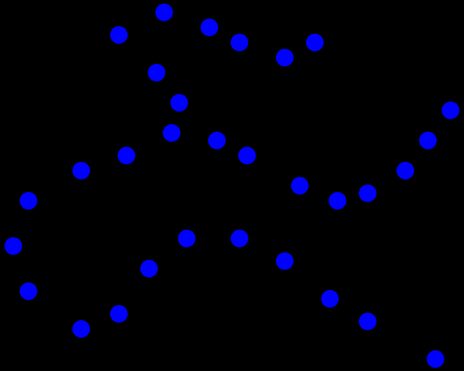


$\dim(H_0) = 31$

Example: expanding balls



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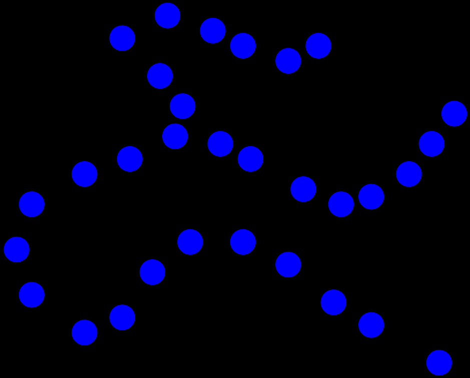

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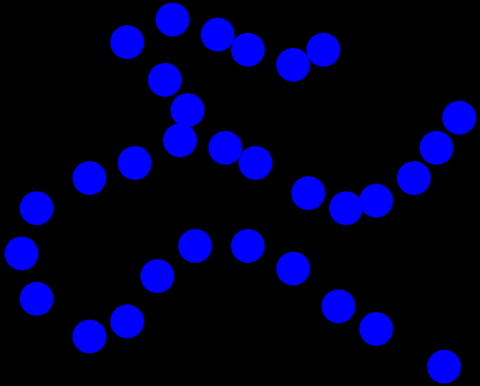
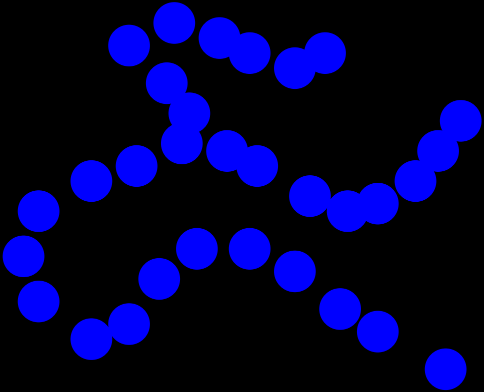


$\dim(H_0) = 26$

Example: expanding balls



$\dim(H_0) = 21$

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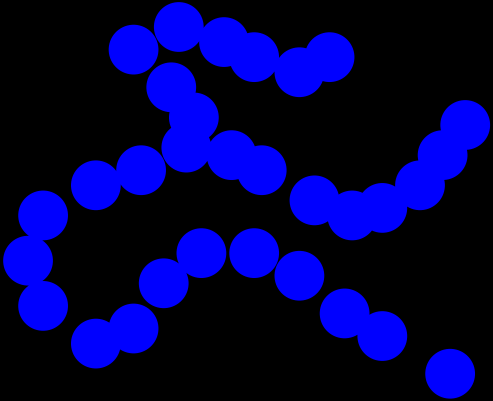
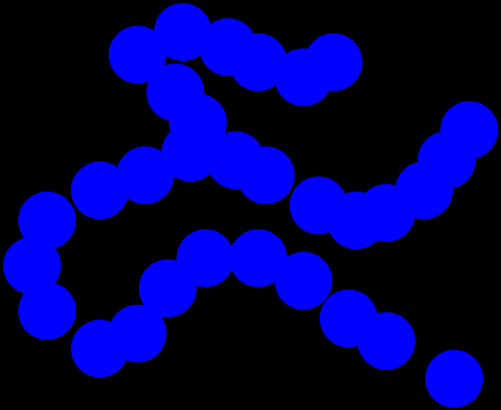


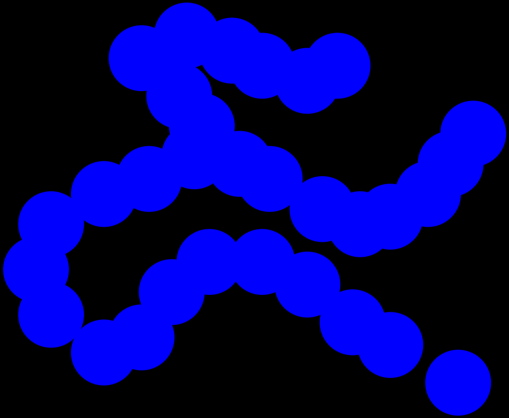
$\dim(H_0) = 12$

Example: expanding balls



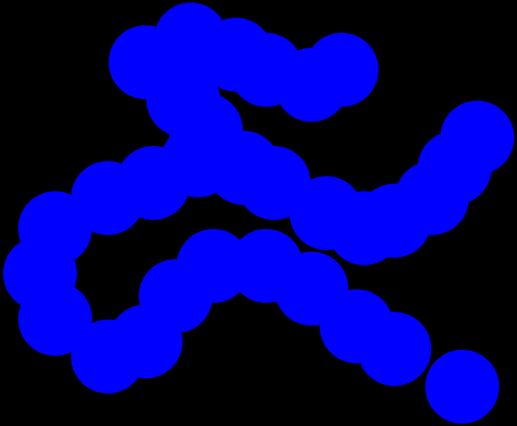
$\dim(H_0) = 6$

Example: expanding balls



$\dim(H_0) = 2$

Example: expanding balls



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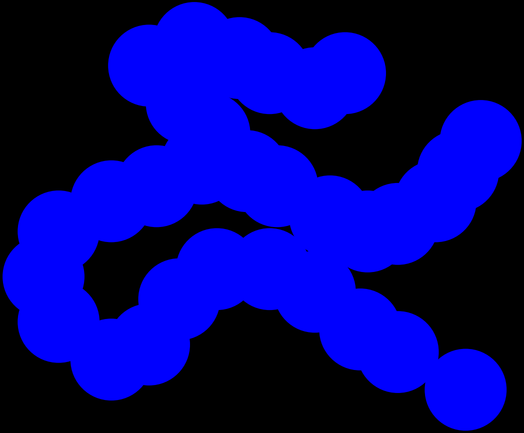
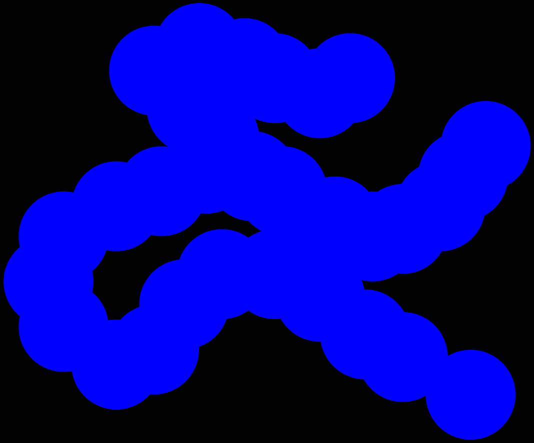


$\dim(H_0) = 1 \qquad \dim(H_1) = 2$

Example: expanding balls



$\dim(H_0) = 1 \qquad \dim(H_1) = 1$

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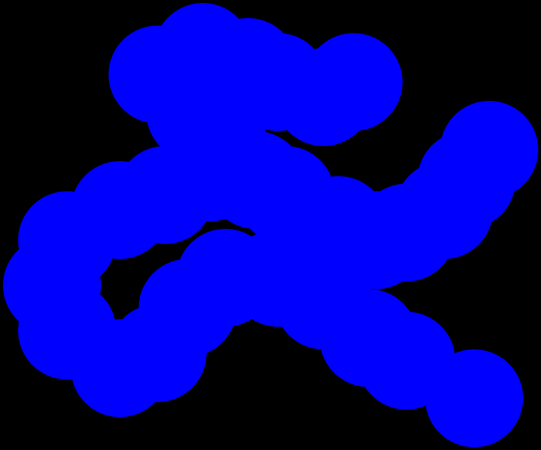

$\dim(H_0) = 1 \qquad \dim(H_1) = 1$

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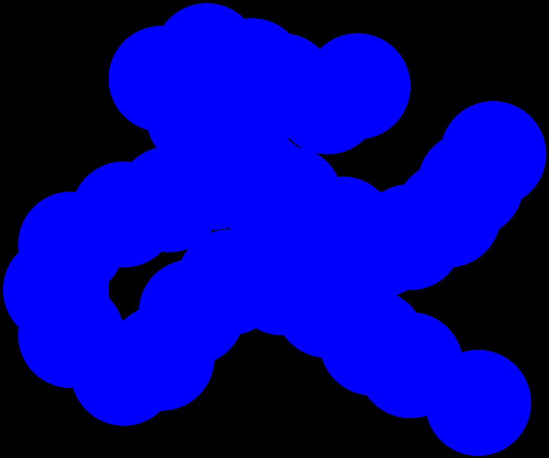
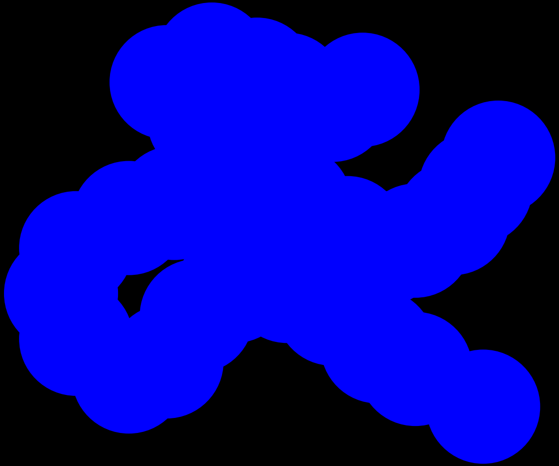


$\dim(H_0) = 1 \qquad \dim(H_1) = 3$

Example: expanding balls



$\dim(H_0) = 1 \qquad \dim(H_1) = 1$

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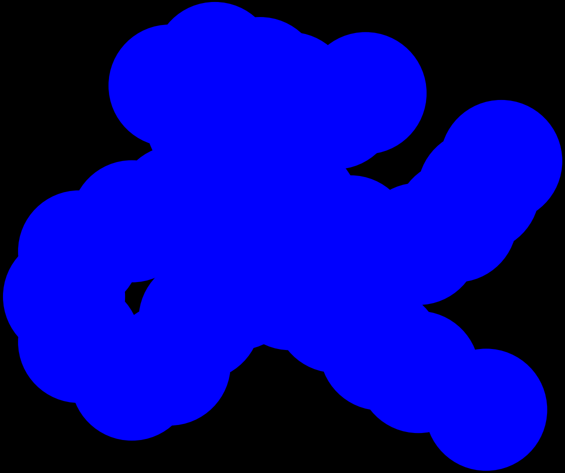

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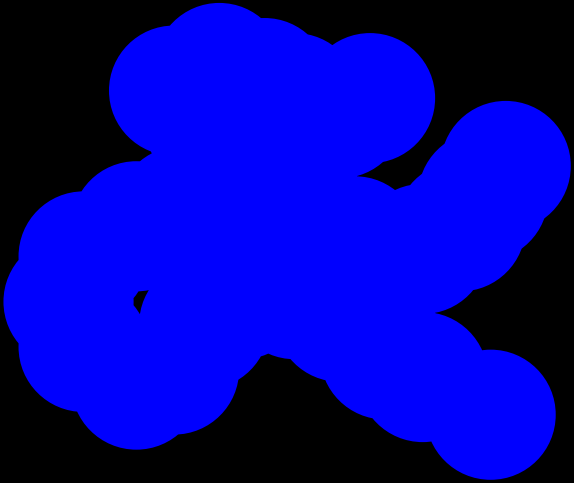

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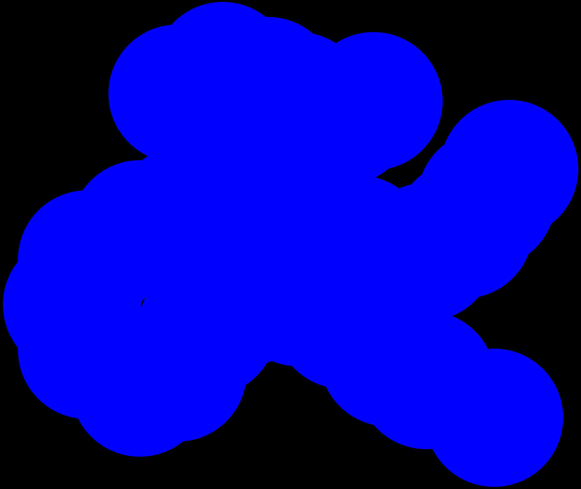

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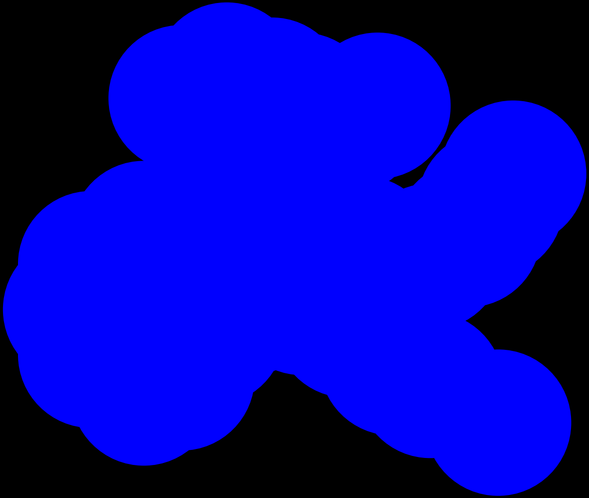

$\dim(H_0) = 1 \qquad \dim(H_1) = 0$

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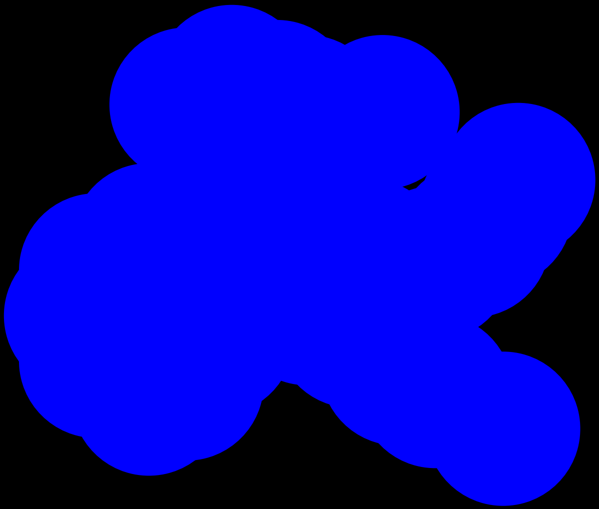

$\dim(H_0) = 1$ $\dim(H_1) = 1$

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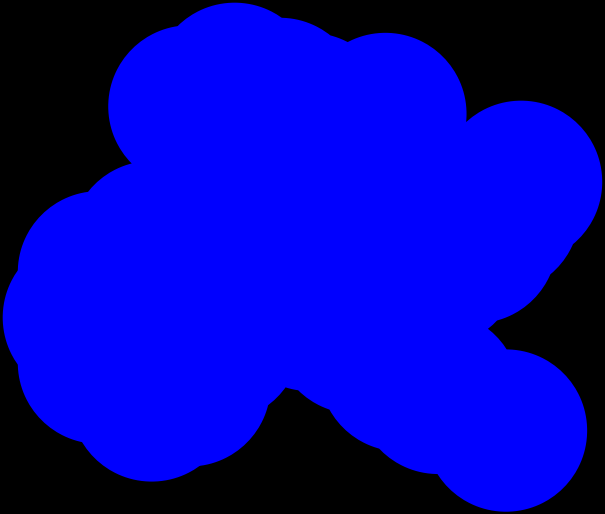

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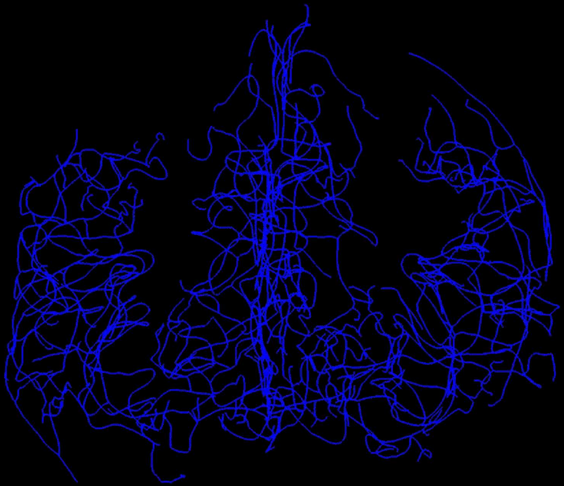

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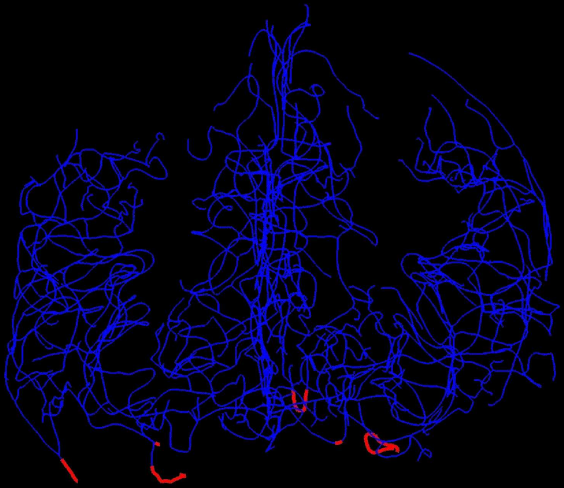

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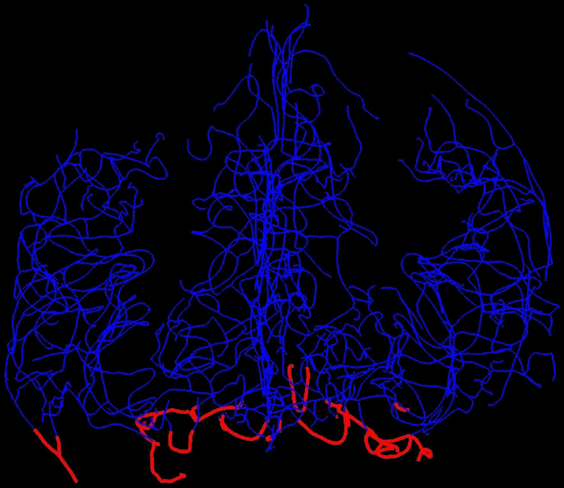

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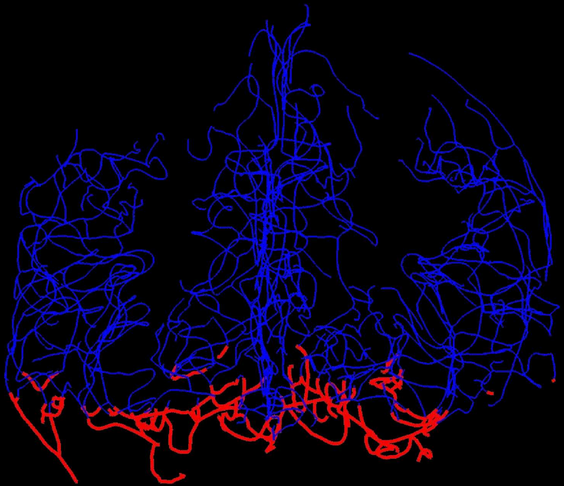

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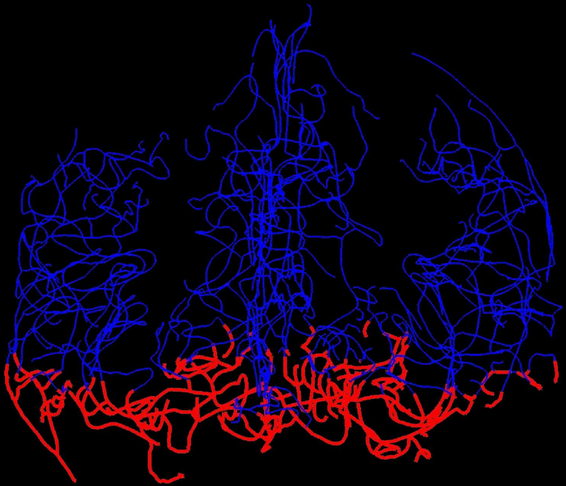

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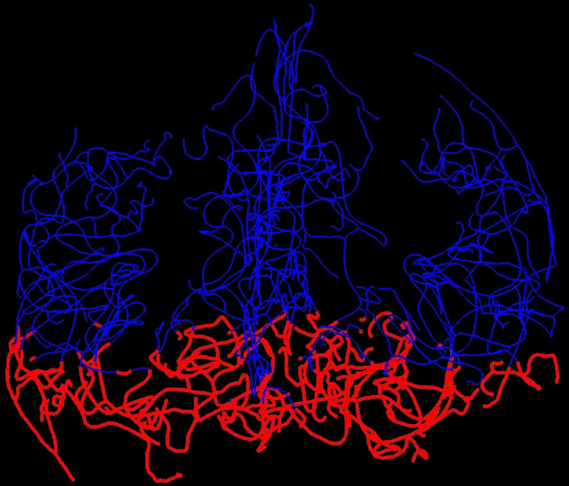

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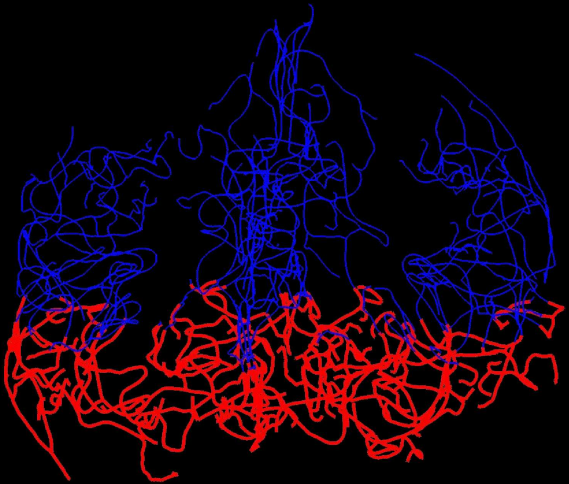

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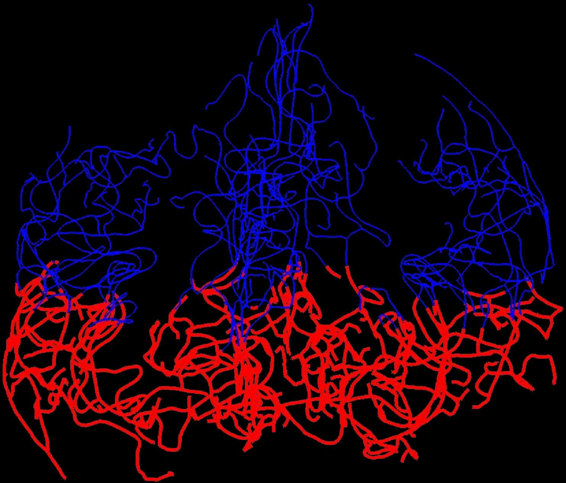

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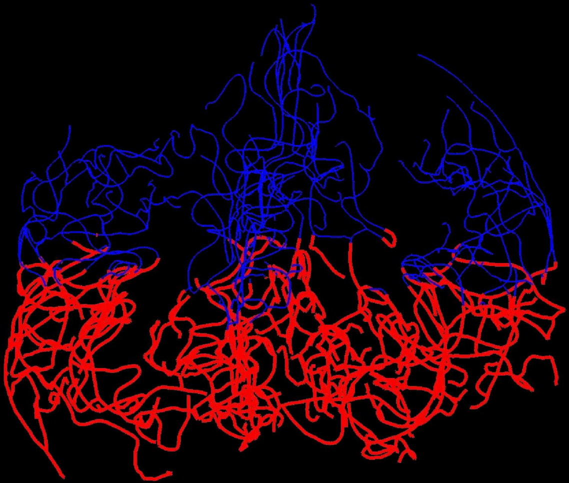

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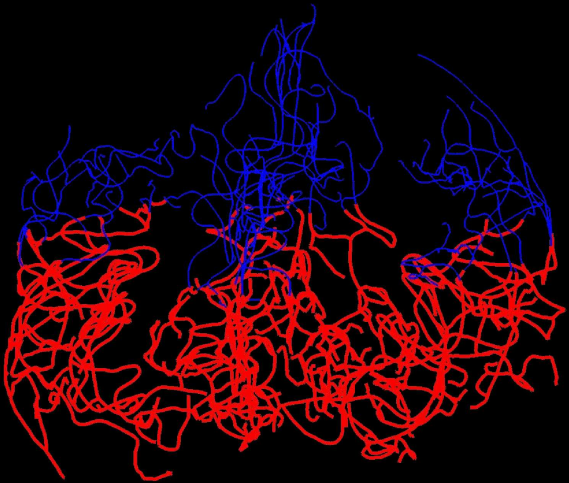

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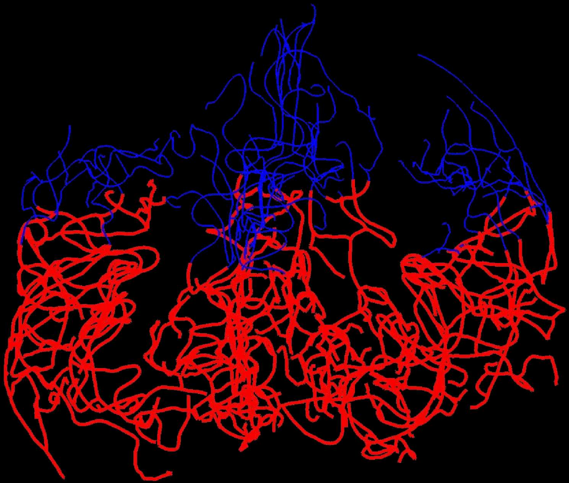

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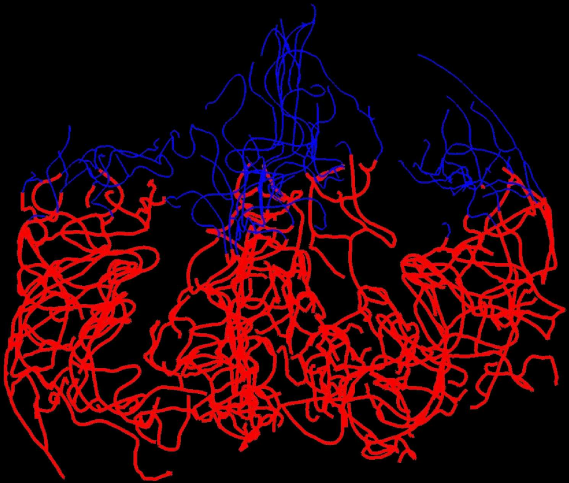

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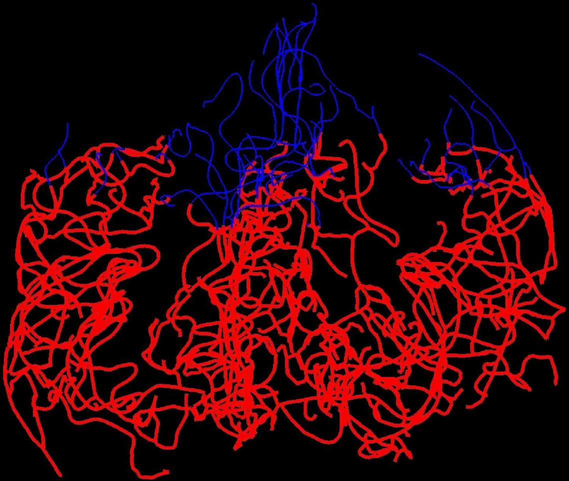

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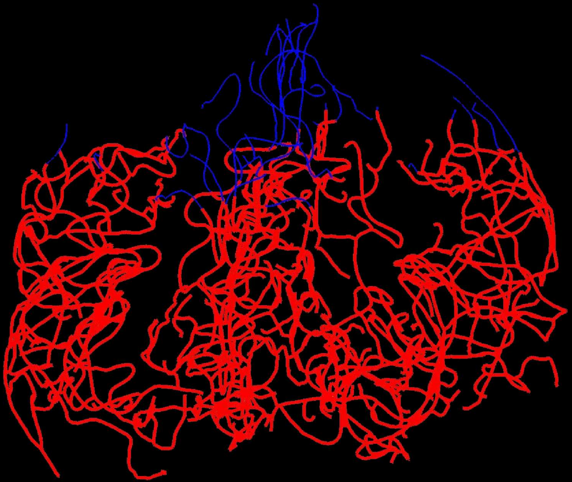

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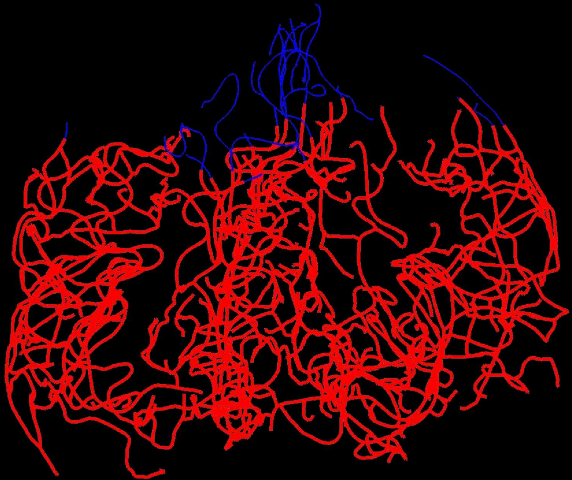

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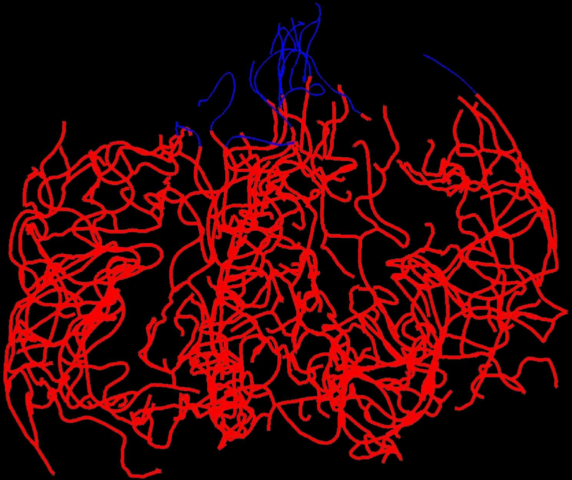

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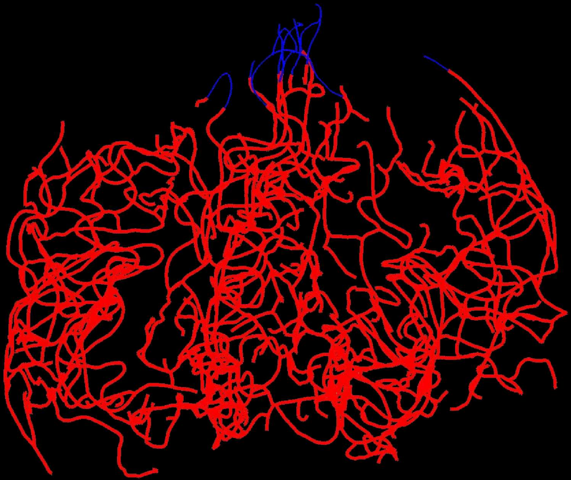

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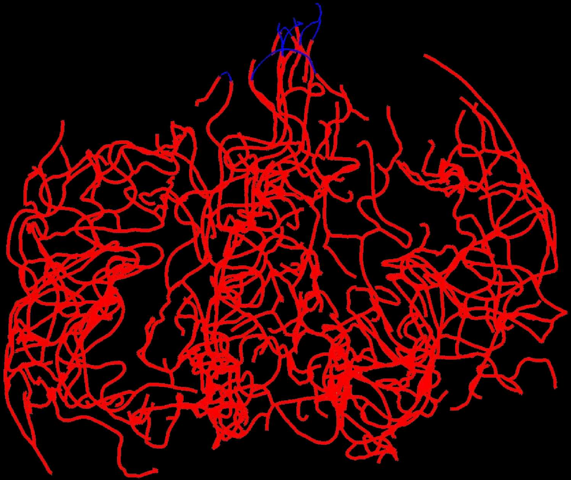

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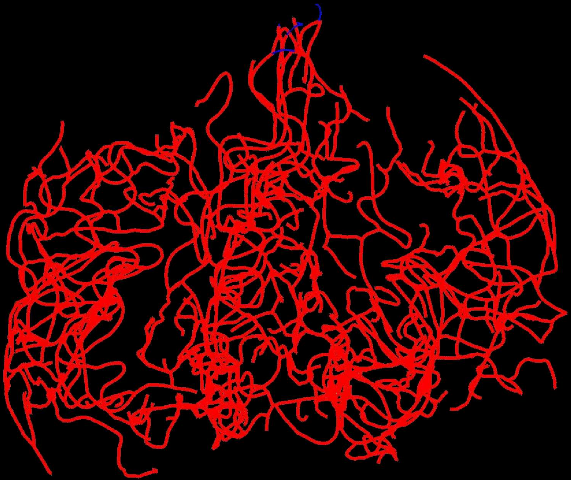

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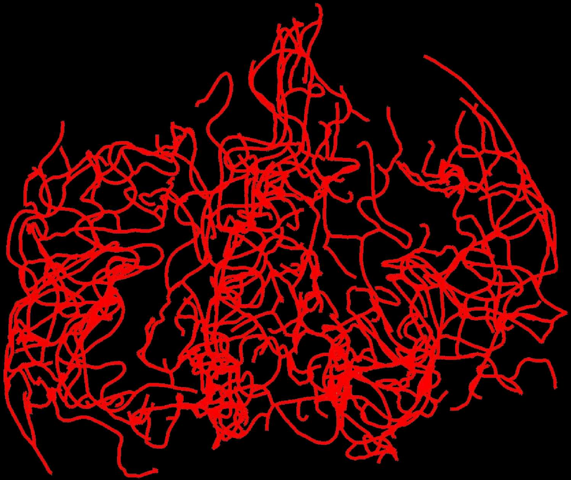

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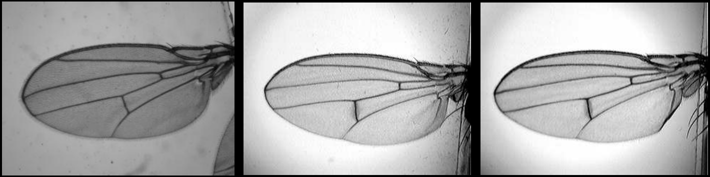
Fruit fly wings

Normal fly wings [images from David Houle's lab]:



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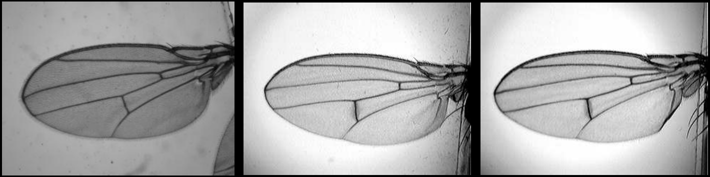
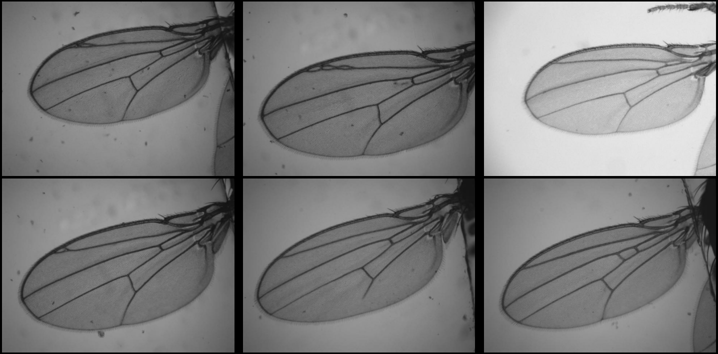
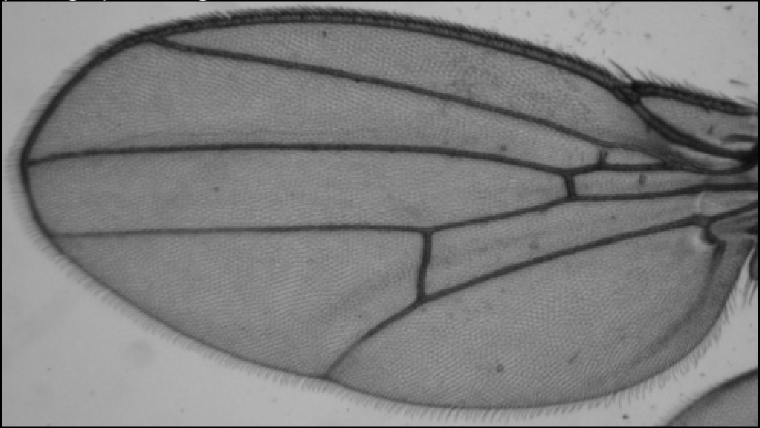


Topologically abnormal veins:



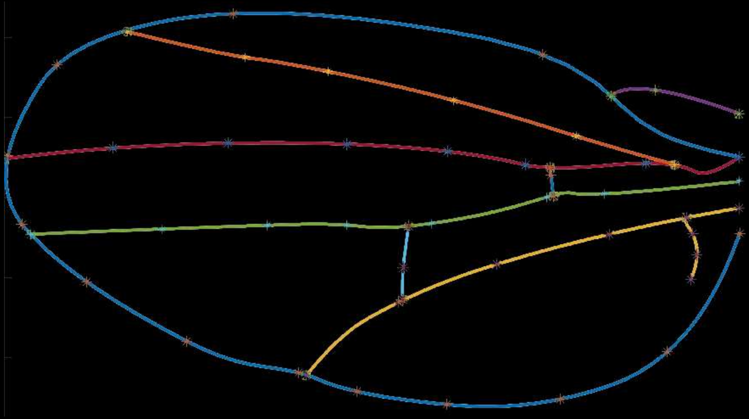
Fruit fly wings

photographic image



Fruit fly wings

spline



Biological background

What generates topological novelty?

[Houle, et al.]: selecting for certain continuous wing vein deformations yields

- skew toward more oddly shaped wings, but also
- much higher rate of topological novelty

Hypothesis. Topological novelty arises when directional selection pushes continuous variation in a developmental program beyond a certain threshold.

Test the hypothesis

- “plot” wings in “form space”
- determine whether topological variants lie “in the direction of” continuous shape selected for, and at the extreme in that direction

Goal. Statistical analysis encompassing topological vein variation, giving appropriate weight to new singular points in addition to varying shape

- compare phenotypic distance to genotypic distance; needs
- metric specifying distance between topologically distinct wings

To proceed. Statistics with fly wings as data objects $\rightsquigarrow$ statistics with multiparameter persistence diagrams as data objects

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Wing vein persistence [w/Houle, et al., ongoing]

Example. Encode fruit fly wing with 2-parameter persistence

- 1st parameter: distance from vertex set
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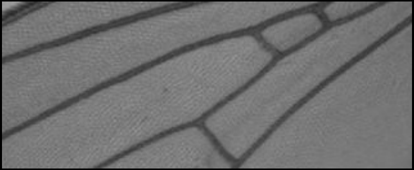

Sublevel set $W_{r,s}$ is near edges but far from vertices $\Rightarrow H_{r,s} = H_i(W_{r,s})$

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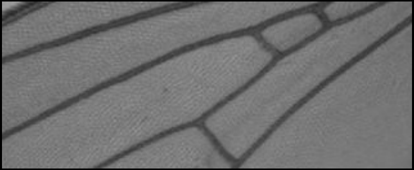

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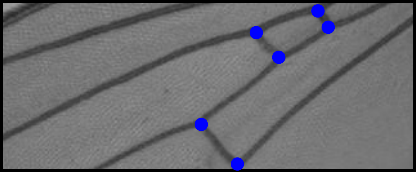

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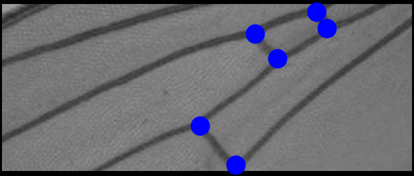

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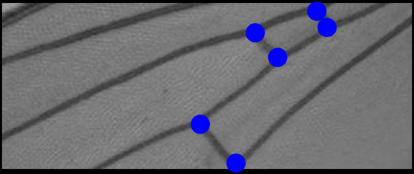

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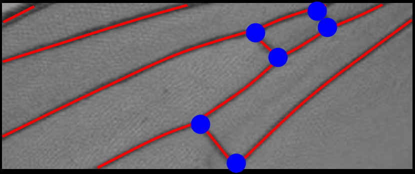

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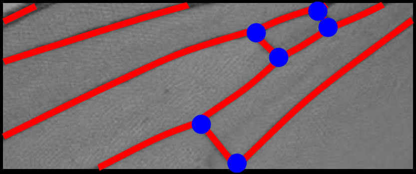

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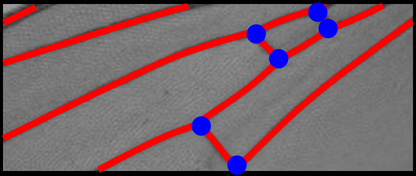

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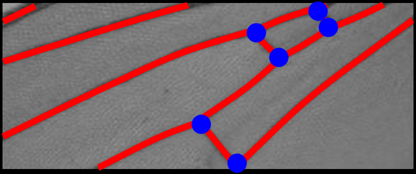

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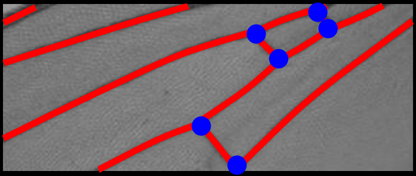

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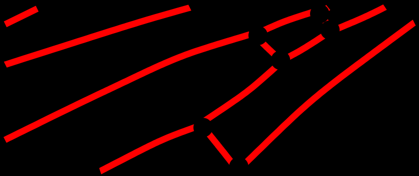

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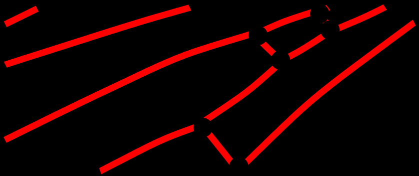

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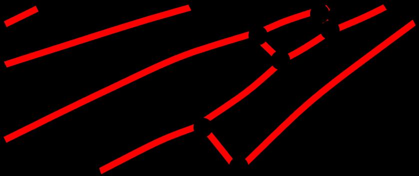

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Multiscale summary

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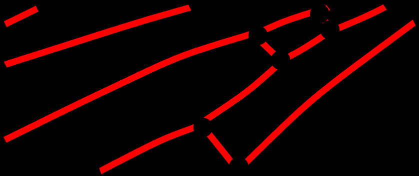

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Multiscale summary

$\mathbb{Z}^2$-module:

$$\begin{array}{ccccccc}
 & \uparrow & & \uparrow & & \uparrow & \\
\to & H_{r-\varepsilon,s+\delta} & \to & H_{r,s+\delta} & \to & H_{r+\varepsilon,s+\delta} & \to \\
 & \uparrow & & \uparrow & & \uparrow & \\
\to & H_{r-\varepsilon,s} & \to & H_{r,s} & \to & H_{r+\varepsilon,s} & \to \\
 & \uparrow & & \uparrow & & \uparrow & \\
\to & H_{r-\varepsilon,s-\delta} & \to & H_{r,s-\delta} & \to & H_{r+\varepsilon,s-\delta} & \to \\
 & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

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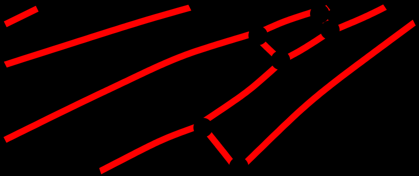

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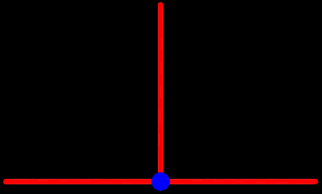
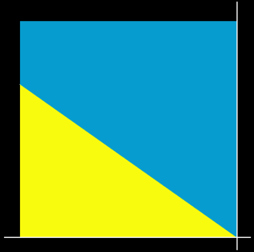


A piece of fly wing vein

$\rightsquigarrow$



The (r, s)-plane $\mathbb{R}^2$

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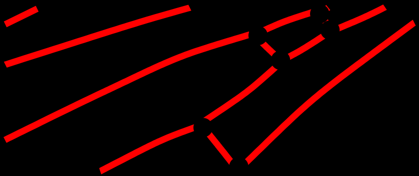

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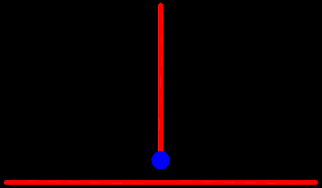

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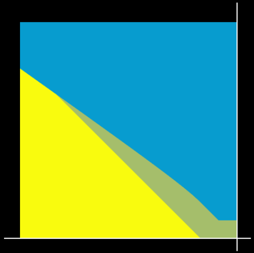

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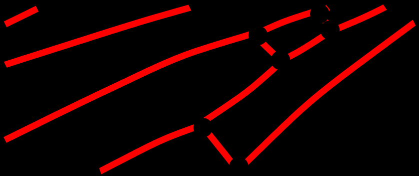

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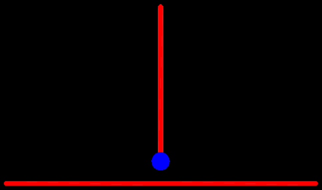

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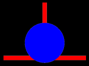

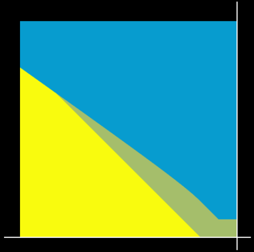

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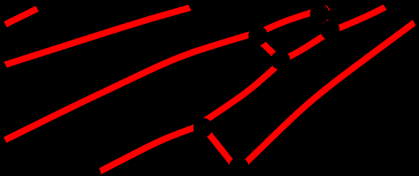

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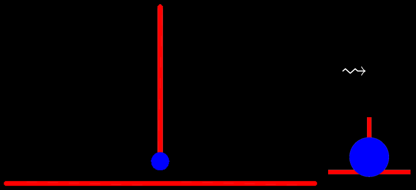

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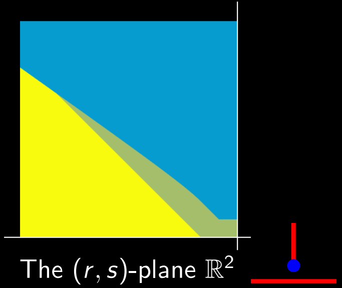

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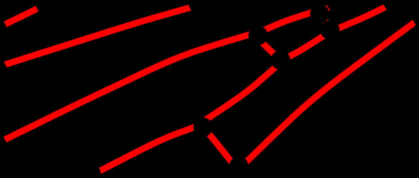

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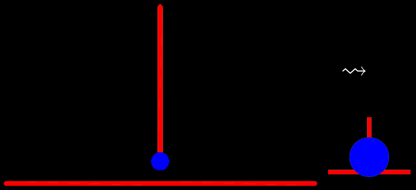

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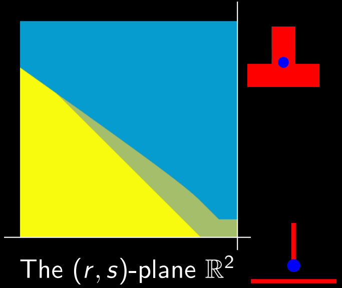

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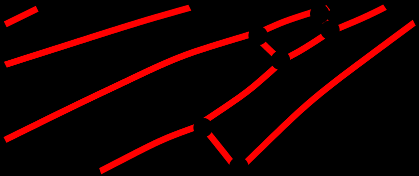

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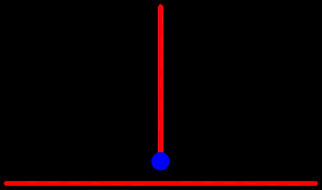

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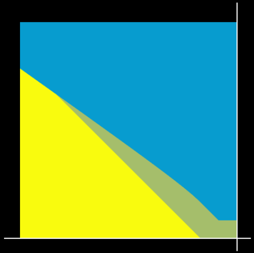

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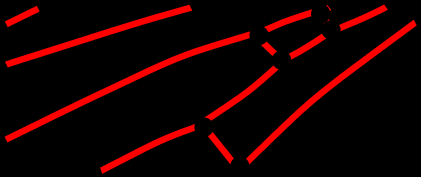

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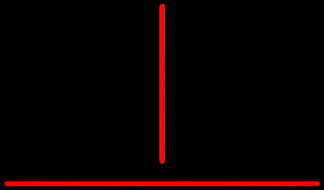

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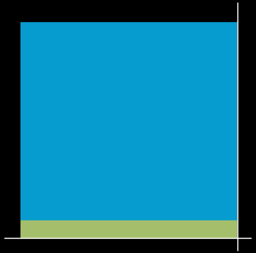

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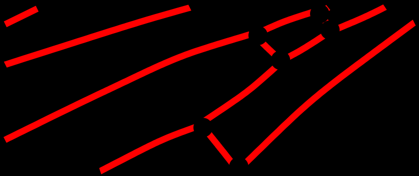

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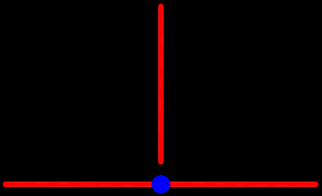
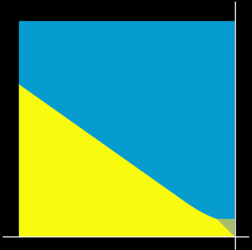


A piece of fly wing vein

$\rightsquigarrow$



The (r, s)-plane $\mathbb{R}^2$

Wing vein persistence [w/Houle, et al., ongoing]

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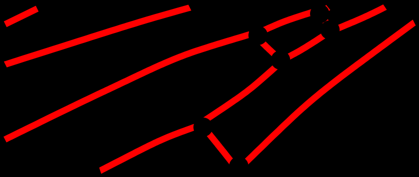

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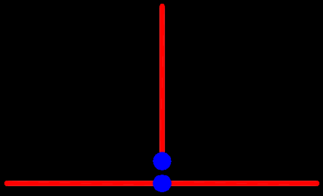

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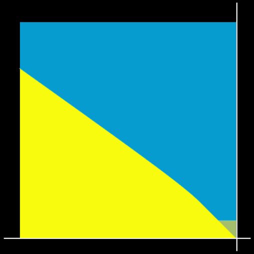

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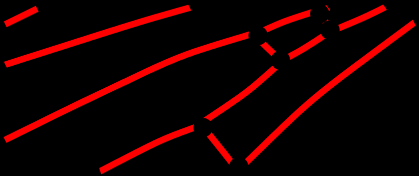

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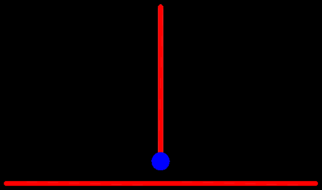

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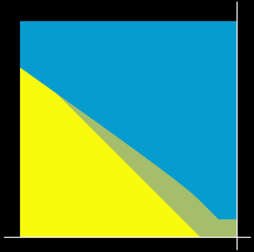

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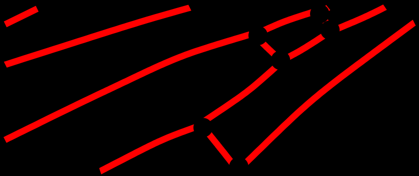

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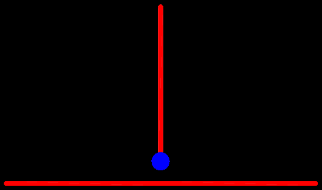

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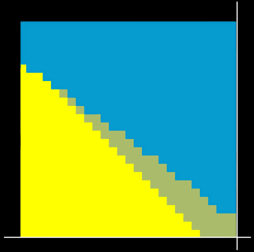

discretized

Persistent homology

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$
$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \mathbb{k})\}_{q \in Q}$. This is a

Def. Q-module over the poset Q:

- family $H = \{H_q\}_{q \in Q}$ of vector spaces over the field $\mathbb{k}$ with
- homomorphism $H_q \to H_{q'}$ whenever $q \prec q'$ in Q such that
- $H_q \to H_{q''}$ equals the composite $H_q \to H_{q'} \to H_{q''}$ whenever $q \prec q' \prec q''$

Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ — 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ — 2 discrete or continuous parameters
- probability distributions: $Q = \mathbb{R}^2$ — 2 continuous parameters
- $Q = \mathbb{Z}^n \Leftrightarrow H = \mathbb{Z}^n$-graded $\mathbb{k}[x_1, \ldots, x_n]$-module
- $Q = \mathbb{R}^n \Leftrightarrow H = \mathbb{R}^n$-graded $\mathbb{k}[\mathbb{R}^n_+]$-module

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Given probability measure μ on a space M and kernel function of bandwidth r

e.g. • K_r = Gaussian (normal distribution) of variance r on $\mathbb{R}^d$

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Def. Convolution with kernel K_r yields bandwidth r expansion $B_r(\mu) = K_r * \mu$.

Example. • $B_r(\mu_n) \sim B_r(\mu)$ if μ_n is uniform on an n-sample from μ

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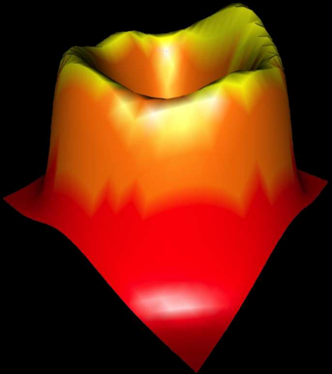

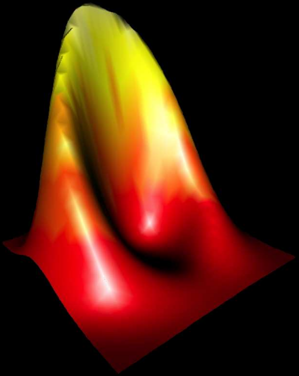

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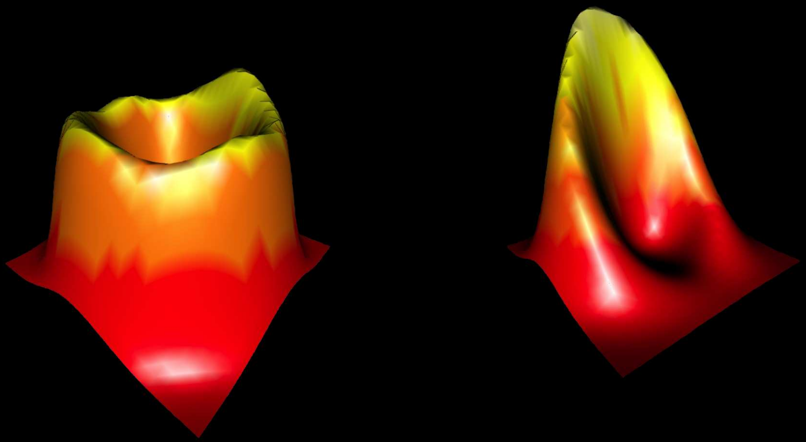

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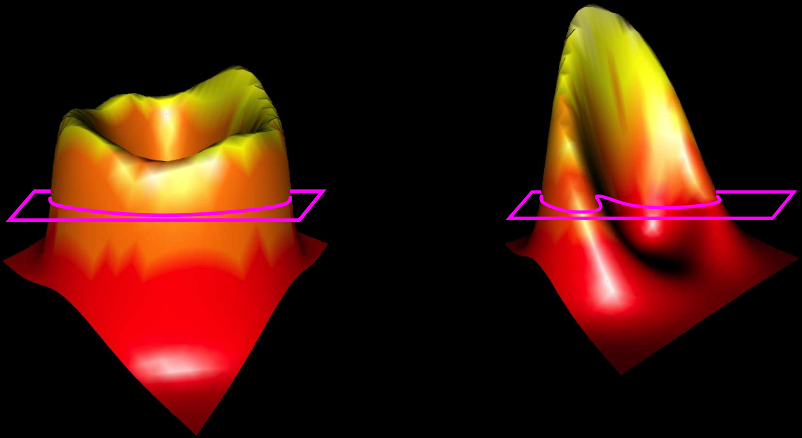

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Def. μ has i^{th} bipersistent homology $H_i^{rs}(\mu) = H_i\big(B_r(\mu)_{r^d s}\big)$, an invariant of μ

$\underbrace{\text{algebra, geometry, combinatorics}}_{\text{presentation, primary decomposition, finite encoding}}$ of $H_*^{rs}(\nu) \leftrightarrow$ statistics of ν

Persistent homology

Input. Topological space X filtered by set Q of subspaces: $X_q \subseteq X$ for $q \in Q$

$\Rightarrow Q$ is a partially ordered set: $X_q \subseteq X_{q'} \Leftrightarrow q \preceq q'$

Def. $\{X_q\}_{q \in Q}$ has persistent homology $\{H_q = H(X_q; \Bbbk)\}_{q \in Q}$. This is a

Def. Q-module over the poset Q:

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Examples

- points in $\mathbb{R}^n$: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- brain arteries: $Q = \{0, \ldots, m\}$ or $\mathbb{R}$ 1-parameter ("ordinary") persistence
- wing veins: $Q = \mathbb{Z}^2$ or $\mathbb{R}^2$ 2 discrete or continuous parameters
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Multigradings

Def. A polynomial in n variables over a field $\Bbbk$ is a (finite) linear combination

$$p(x_1,\dots,x_n) = \sum_{\mathbf{a}\in\mathbb{N}^n} c_\mathbf{a}\mathbf{x}^\mathbf{a} \qquad \text{with} \qquad \mathbf{x}^\mathbf{a} = x_1^{a_1}\cdots x_n^{a_n}.$$

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2. monomial ideal $I \subseteq \Bbbk[Q]$

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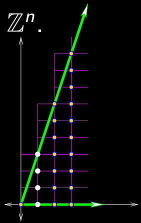

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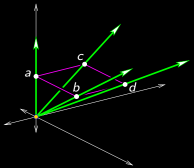

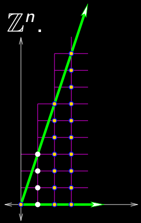

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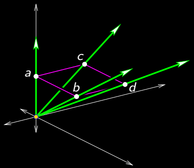

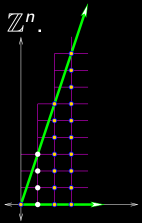

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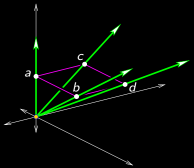

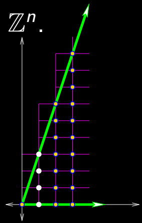

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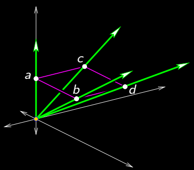

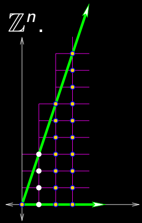

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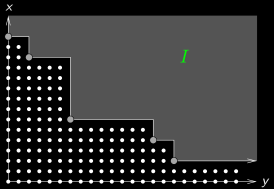

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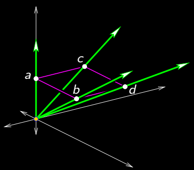

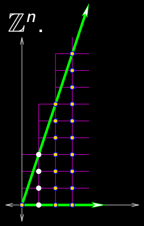

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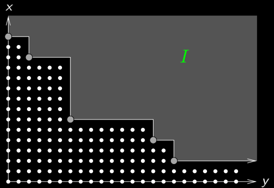
2. monomial ideal $I \subseteq \Bbbk[Q]$
3. quotient $\Bbbk[Q]/I$



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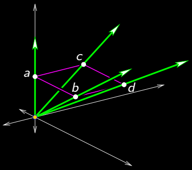

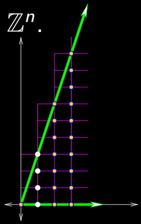

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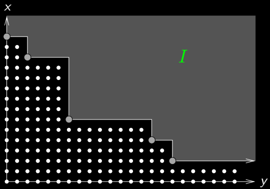

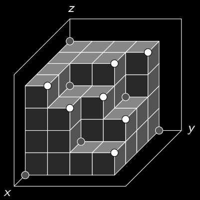

Polynomials with real exponents

Def. A real-exponent polynomial in n variables over $\Bbbk$ is a linear combination

$$p(x_1, \ldots, x_n) = \sum_{\mathbf{a} \in \mathbb{R}^n_+} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \qquad \text{with} \qquad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}.$$

The set of these is the real-exponent polynomial ring $\Bbbk[\mathbb{R}^n_+]$, with $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$.

Example. $(x^{\sqrt{3}} + y^{\pi})(xy^2 - z) = x^{1+\sqrt{3}}y^2 + xy^{2+\pi} - x^{\sqrt{3}}z - y^{\pi}z$

Def. A multigraded module over $\Bbbk[\mathbb{R}^n_+]$ is

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Remark. Real-exponent polynomials behave poorly.

1. $I = \langle x_1^{a_1}, \ldots, x_n^{a_n} \mid a_i > 0 \ \forall\ i \rangle = \mathfrak{m}$ = maximal monomial ideal
   - countably generated
   - no minimal generating set
2. $I = \langle \mathbf{x}^{\mathbf{a}} \mid a_1 + \cdots + a_n = 1 \text{ and } a_i \geq 0 \ \forall\ i \rangle$
   - uncountably generated
   - unique minimal monomial generating set

In $\Bbbk[\mathbf{x}]$: all ideals finitely generated.

Polynomials with real exponents

Def. A real-exponent polynomial in n variables over $\Bbbk$ is a linear combination

$$p(x_1, \ldots, x_n) = \sum_{\mathbf{a} \in \mathbb{R}_+^n} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \qquad \text{with} \qquad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}.$$

The set of these is the real-exponent polynomial ring $\Bbbk[\mathbb{R}_+^n]$, with $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$.

Example. $(x^{\sqrt{3}} + y^{\pi})(xy^2 - z) = x^{1+\sqrt{3}}y^2 + xy^{2+\pi} - x^{\sqrt{3}}z - y^{\pi}z$

Def. A multigraded module over $\Bbbk[\mathbb{R}_+^n]$ is

$$M = \bigoplus_{\mathbf{b} \in \mathbb{R}^n} M_{\mathbf{b}} \qquad \text{with action} \qquad \mathbf{x}^{\mathbf{a}} M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}.$$

Remark. Real-exponent polynomials behave poorly.

1. $I = \langle x_1^{a_1}, \ldots, x_n^{a_n} \mid a_i > 0 \ \forall\ i \rangle = \mathfrak{m} =$ maximal monomial ideal
   - countably generated
   - no minimal generating set
2. $I = \langle \mathbf{x}^{\mathbf{a}} \mid a_1 + \cdots + a_n = 1 \text{ and } a_i \geq 0 \ \forall\ i \rangle$
   - uncountably generated
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In $\Bbbk[\mathbf{x}]$: all ideals finitely generated.

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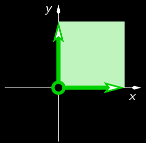

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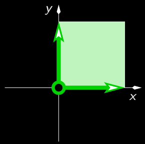

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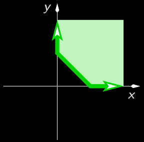

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History of persistent homology

Ordinary persistence

- traces back to [Morse 1940s]
- bar codes [Abeasis–Del Fra 1980], rediscovered many times
- formally defined [Frosini, Landi 1999], [Robins 1999]
- efficient computation [Edelsbrunner, Letscher, Zomorodian 2002]
- applications [too many to list; a few early ones, but most roughly 2013–]

Multiparameter persistence

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    - + usually assume finitely presented, even if over $\mathbb{R}^n$

Essentially equivalent

- representation of Q [Nazarova–Roiter 1972]
- functor from Q to the category of vector spaces (e.g., [Curry 2019])
- vector-space valued sheaf on Q (e.g., [Yuzvinsky 1987], [Yanagawa 2001], [Curry 2014])
- representation of incidence algebra of Q [Doubilet–Rota–Stanley 1972]
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Where are we going?

II. Finiteness conditions

- tame
- finitely encoded

III. Presentation and resolution

- generators and cogenerators
- presentation: free, injective, upset, downset, fringe
- resolution: free, injective, upset, downset
- tameness criteria: syzygy theorem

IV. Data structures: monomial matrices

- Hom sets
- proof of tameness criteria

V. Measures of size and distances between persistence modules

- rank and Hilbert function
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- transience and persistence along faces
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