Realizability questions in topology: the group of self homotopy equivalences and permutation modules

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Realisability questions are a classical branch of problems in Algebraic Topology. Perhaps the most notable example is the realisability of cohomological algebras which was proposed by N. E. Steenrod in the 1960s. Also in the 1960s, D. Kahn [Kah] asked if one could realise any group as the group of self homotopy equivalences of a simply connected space, in what became known as the *realizability problem for abstract groups*. In 2014, an answer in the positive was given by Costoya and Viruel for the case of finite groups [CV]. In 2021, Chocano, Morón and Ruiz del Portal used Alexandroff spaces to answer the full general case at the cost of dropping the simply connected assumption [CMR]. However, the general (simply-connected) case is still an open problem.

In this talk, I will explain some techniques to tackle realizability questions in the context of topological spaces. In particular, we will discuss the tecnhiques developed by Costoya and Viruel, which combine a theorem of Frucht, that says that every group is the group of automorphisms of some simple graph, with rational homotopy techniques to realize every finite group as the group of self homotopy equivalences of an elliptic space. Finally, I will share the main ideas of a more recent work in which we realize permutation modules as the integral homology of a finite space. This is a refinement of the *realizability problem for abstract groups*, as it involves realizing the group of self homotopy equivalences and its natural action on the homology groups. For the same reason, it also extends the work of Chocano, Morón and Ruiz del Portal.

This is joint work with Cristina Costoya and Antonio Viruel

References

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