# Stability of Optical Solitons in 2d

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**Abstract**

Optical properties of nematic liquid crystals have received great attention in the last years, as they can support stationary optical waves. Due to its high susceptibility, the response of a nematic liquid crystal to a light beam propagating through it is nonlocal and nonlinear. This response has a self-focusing effect on the light beam, supporting waveguides that counterbalance the diffraction spreading nature of light beam, and, in optimal shapes, allows the existence of stationary waves.

We study the ground states of the Schrödinger-Poisson system in dimension $(2+1)$

$$i\partial_z u + \frac{1}{2}\Delta u + u\sin(2\theta) = 0 \tag{1}$$

$$-\nu\Delta\theta + q\sin(2\theta) = 2|u|^2\cos(2\theta) \tag{2}$$

that models the propagation of a laser beam through a planar cell filled with a nematic liquid crystal.

The axis $z$, referred to as the optical axis, is the direction of the propagation of a light beam, while $\Delta$ is the Laplacian in the transverse coordinates $(x, y)$. Equation (1) represents the evolution of the light beam, with $u : \mathbb{R}^2 \to \mathbb{C}$ the complex amplitude of the electric field, while (2) is the nonlocal response of the medium, with $\theta : \mathbb{R}^2 \to \mathbb{R}$ the director field angle of the light-induced reorientation of crystal liquid molecules; $q, \nu$ are positive constants depending on the physics of the experiment.

In [2] a heuristic derivation of the equations is presented in the Appendix, while [1] and references therein give a more precise physical derivation.

The system was rigorously studied in [2], where the authors proved global existence and regularity for the Cauchy problem, and existence of stationary waves as minimizers, over the couples $(u, \theta) \in H^1 \times H^1$ with $L^2$ norm of $u$ fixed, of the Hamiltonian:

$$E(u,\theta) := \frac{1}{4}\int_{\mathbb{R}^2} |\nabla u|^2 + \nu|\nabla\theta|^2 - 2|u|^2\sin(2\theta) + q(1-\cos(2\theta))\,dx \tag{3}$$

We present a first stability result for those stationary waves. This provides a strong justification of the relevance of the mathematical model to applications, as only locally stable solutions are expected to be seen in experiments and numerical simulations. Our main result is

**Theorem 1.** *Let $(v, \phi)$ be the configuration of minimal energy $E$ over the constraint*

$$S_a := \left\{(u, \theta) \in H^1 \times H^1 \mid \|u\|^2_{L^2} = a\right\}$$

*Then $(v, \phi)$ is orbitally stable with respect to the evolution* (1)-(2).

As a definition of orbital stability we ask, loosely speaking, that the evolution through equations (1)-(2) of an initial datum close to $v$ in $H^1$ remains close, modulo the symmetries of the Hamiltonian, to the ground state for all times.

In our proof, we adapt to the coupled system the arguments of [3], where stability is obtained from the positivity of the second derivative of the action. We prove at first a new estimate for perturbed configurations close to the ground state, showing that perturbation of the angle $\theta$ are controlled in norm by the associated perturbation of the light beam, up to a constant that depends on the shape of the ground state. This allows to focus the stability study mainly on to the variable $u$. Hence we show positivity explicitly by Taylor expansion and minimization properties of the ground state.

# References


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[3] M. Grillakis, J. Shatah, W. Strauss : *Stability theory of solitary waves in the presence of symmetry, I* Journal of Functional Analysis 74, (1987)