Anomalous diffusion and homogenization

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We construct an incompressible, time-dependent vector field $\mathbf{b}(t, x)$ which is Holder continuous up to exponent 1/3, for which the associated passive scalar (or advection-diffusion) equation,

$$\partial_t \theta^\kappa - \kappa \Delta \theta^\kappa + \mathbf{b} \cdot \nabla \theta^\kappa = 0$$

exhibits anomalous diffusion. This means that the L^2 oscillation of the solution θ^κ is dissipated in unit time, even if the diffusivity parameter κ is approaching zero. Such behavior is predicted to occur for a fully turbulent velocity field. We also propose an analytical framework in which to study anomalous diffusion, via a joint forward cascade of energy and a backward cascade of renormalized eddy viscosities. The idea is to coarse-grain the system, and study the "effective diffusivity" as a function of the (logarithm of the) length scale. We show that the length scales interact in a "reverse cascade" as we zoom out (go up the scales) and cause the effective diffusivity to increase, which leads to the result. The proof is formalized using quantitative homogenization estimates, iterated across infinitely many scales, which is made possible by carefully designing the vector field. This is joint work with Vlad Vicol.