A Subcritical uncertainty principle for the Fourier Transform

The uncertainty principle relations forbid a non-zero pair $(f, \hat{f})$ from being "too" (or critically) localized. There are many instances of this type results, such as the Benedicks uncertainty principle, which says that f and $\hat{f}$ cannot both be compactly supported unless $f \equiv 0$, and the Heisenberg uncertainty principle, which can be written as the following inequality for $f \in L^2$: $\|f(x)\|_2 \lesssim \|f(x)|x|\|_2 + \|\hat{f}(\xi)|\xi|\|_2$.

In this talk, the results that we will explore are related to Hardy's and Cowling-Price's uncertainty principles, which assert that for any pair $(f, \hat{f})$ which satisfies $f(x)e^{\pi x^2}, \hat{f}(\xi)e^{\pi \xi^2} \in L_p$ one has $f(x) = Ce^{-\pi x^2}$ if $p = \infty$ and $f \equiv 0$ if $1 \le p < \infty$.

In 2008, Vemuri initiated the study of subcritical pairs in the sense of Hardy, that is, those $(f, \hat{f})$ which satisfy $f(x)e^{(\pi-\varepsilon)x^2}, \hat{f}(\xi)e^{(\pi-\varepsilon)\xi^2} \in L_\infty$ for $\varepsilon > 0$ very small, and, in connection with the Schrödinger equation, conjectured that such functions must necessarily exhibit some strong properties. This conjecture has been recently settled by Radchenko and Ramos.

Here we will study the properties of subcritical pairs in the sense of Cowling-Price. More precisely, we will discuss the sharp rate of pointwise decay of pairs $(f, \hat{f})$ which satisfy $f(x)e^{(\pi-\varepsilon)x^2}, \hat{f}(\xi)e^{(\pi-\varepsilon)\xi^2} \in L_p$ for $p < \infty$. This work is related to the recent approach by Kulikov, Oliveira and Ramos to tackle the previous conjecture by Vemuri through estimates for the Schrödinger equation.

Our approach relies on comparing the pointwise value of $f(x)$ with a suitable average around x and then applying techniques from Approximation Theory.

Joint work with Sergey Tikhonov.