On the number of multidimensional integer partitions

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The partition function $p_2(n)$ that counts the number of integer partitions of a positive integer n, i.e. the number of ways to write n as a sum of nonincreasing positive integers, is a very well studied object, which apart from having many important and beautiful properties is known to have the asymptotics $p_2(n) \sim \exp(\pi\sqrt{2n/3})(4\sqrt{3}n)^{-1}$. Importantly, any integer partition $n = n_1 + n_2 + ...n_k,\ n_1 \geq n_2 \geq ... \geq n_k$, can be visualized via the Young diagram, which consists of n cells placed in k rows and n_1 columns so that the ith row contains n_i cells and the first cell in each row belongs to the first column.

As for the general multidimensional case, a d-dimensional partition of n is its representation of the form

$$n = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}...\sum_{i_d=1}^{\infty} n_{i_1 i_2 ... i_d}, \quad n_{i_1 i_2 ... i_d} \in \mathbb{Z}_+,$$

where $n_{i_1 i_2 ... i_d} \geq n_{j_1 j_2 ... j_d}$ if $j_k \geq i_k$ for all $k = 1, 2, ..., d$. There is a one-to-one correspondence between $(d-1)$-dimensional partitions and their d-dimensional "geometric visualizations" that we call lower sets.

More precisely, we call a set $S \subset \mathbb{Z}_+^d,\ d \geq 2$, a lower set if for any $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{Z}_+^d$ the condition $\mathbf{x} \in S$ implies $\mathbf{x}' = (x_1', ..., x_d') \in S$ for all $\mathbf{x}' \in \mathbb{Z}_+^d$ with $x_i' \leq x_i,\ 1 \leq i \leq d$. Equivalently, one can think of a d-dimensional lower set as of a union of unit cubes such that in each direction any cube leans either on another one or on the coordinate hyperplane. On the figure below one can see an example of a three-dimensional lower set.

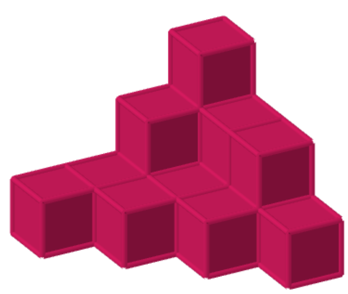

Figure 1.

We are interested in estimating the number $p_d(n)$ of d-dimensional lower sets of cardinality n. The two-sided inequality $n^{1-1/d} < \log p_d(n) < C(d)n^{1-1/d}$ is known to be true whenever $\log n > 3d$. We show that if d is sufficiently small with respect to n, then C does not depend on d, which means that $\log p_d(n)$ is up to an absolute constant equal to $n^{1-1/d}$.