Existence and regularity of the minima of integral shape functionals

Francesco Paolo Maiale

The talk is based on joint work with G. Buttazzo, D. Mazzoleni, G. Tortone and B. Velichkov

Abstract

We consider a shape optimization problem formulated in the optimal control form: the governing operator is the p-Laplacian in $\mathbb{R}^d$, the cost is of an integral type, and the control variable is the domain of the state equation. More precisely, given $D \subset \mathbb{R}^d$, for every $\Omega \subset D$, u_Ω is the solution of

$$-\Delta_p u = f \text{ in } \Omega, \quad u = 0 \text{ on } \overline{D} \setminus \Omega.$$

We consider minimization problems of the form $\min\{J(u_\Omega) : \Omega \text{ open}, \Omega \subset D\}$ for cost functionals of the form

$$J(u_\Omega) = \int_\Omega [-g(x)u_\Omega(x) + Q(x)]\, dx$$

where g and Q are given and satisfy certain assumptions.

The first part of the talk discusses conditions ensuring the existence of an optimal domain. More precisely, we show that optimal domains have finite perimeters and, under certain conditions, are open sets. A key distinction lies in cases where $p > d$ for which existence holds under mild conditions, and cases where $p \leq d$, requiring additional assumptions on the data.

The second part of the talk focuses on exploring the regularity of optimal shapes. We establish the first regularity theorem for the free boundary of solutions to the shape optimization problems above. Despite the challenge that the minimality of a domain Ω cannot be expressed as a variational problem for a single state function, we study the blow-up limits of the state function u_Ω by performing a triple consecutive blow-up, eventually obtaining a sequence converging to a homogeneous stable solution of the one-phase Bernoulli problem. This allows us to decompose $\partial\Omega$ into a singular and a regular part, with the latter showing C^∞-regularity when the data are smooth.

Finally, to estimate the Hausdorff dimension of the singular part, we introduce a new formulation of stability for the one-phase problem, which is preserved under blow-up limits and allows to develop a dimension reduction principle.