Rate-Efficient Asymptotic Normality for the Fourier Estimator of the Leverage Process

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Extended Abstract

The leverage effect, introduced in a seminal paper by [Black, 1976], refers to the relationship between asset price returns and volatility, which tend to be negatively correlated. A flourishing literature has recently confirmed both theoretically and empirically the fact that the leverage effect, interpreted in common stochastic volatility models as the quadratic covariation between the asset log-price and the variance processes, is time dependent and random, see [Bandi and Renò, 2012], [Veraart and Veraart, 2012], [Mykland and Wang, 2014], [Kalnina and Xiu, 2017], [Curato, 2019] to cite few of them. Nonetheless, the estimation of the leverage effect represents a challenging task, because the volatility process is not observable, thus requiring its pre-estimation as a first step.

In this paper we consider two estimators of the (integrated) leverage based on the Fourier covariance estimation method by [Malliavin and Mancino, 2002a] and prove the rate-efficient central limit theorem for these estimators. The Fourier estimation method is particularly suited to build estimators of second-order quantities as the leverage, which is obtained as the covariation between the (estimated) stochastic variance process and the (observed) asset log-price process. In fact, as a first step the Fourier method is applied to obtain estimates of the Fourier coefficients of the volatility. Then, the knowledge of the Fourier coefficients of the latent volatility process allows iterating the procedure in order to compute the Fourier coefficients of the covariance process between the log-price and the volatility (i.e., the leverage). In particular, the integrated leverage requires to compute only the 0-th Fourier coefficient of the covariance process. It is worth noting that this procedure does not require the preliminary estimation of the instantaneous volatility path, but only integrated quantities, namely, the Fourier coefficients of the volatility.

An early attempt to use the Fourier method to identify the parameters of stochastic volatility models is present in [Malliavin and Mancino, 2002b], [Barucci and Mancino, 2010]. [Curato, 2019] proves

the asymptotic error normality for the Fourier estimator of leverage with a rate lower than 1/6. The low rate found in [Curato, 2019] is a consequence of the assumption that the number of frequencies M employed for the second step, namely the convolution product between the Fourier coefficients of the log-price and those of the volatility, satisfies $M^3/n \to 0$, where n is the number of price observations. As the asymptotic rate of the Fourier leverage estimator is $M^{1/2}$, the result found in [Curato, 2019] is clear. On the other side, due to the low rate of convergence, the asymptotic variance does not depend on the number of the Fourier coefficients N of the log-price used for the first step (i.e., the volatility estimation). Further, [Curato and Sanfelici, 2019] study the finite sample properties of the Fourier estimator of the integrated leverage effect in the presence of microstructure noise contamination, showing its asymptotic unbiasedness under this condition.

In the present paper we prove that, in the continuous stochastic volatility model considered in [Curato, 2019], a careful choice of the two cutting frequencies which define the Fourier leverage estimator allows reaching the optimal rate 1/4. We show that the asymptotic variance depends on both the frequencies M and N, except in the case where N is chosen to be the Nyquist frequency $n/2$, which is the natural choice in the absence of microstructure noise, indeed. Furthermore, we consider two different convolution products for the second step: the first one by means of the Dirichlet kernel and the second one using the Fejer kernel. As it is well known, this choice does not affect the rate of convergence, but the asymptotic variance. Both the Fourier leverage estimators reach a smaller asymptotic variance with respect to the leverage estimator in [Mykland and Wang, 2014], while only the Fourier leverage estimator with the convolution obtained using the Fejer kernel has a smaller asymptotic variance with respect to the leverage estimator in [Aït-Sahalia and Jacod, 2014]. The leverage estimator in [Aït-Sahalia et al., 2017] reaches an even smaller asymptotic error variance.

The analytical results are corroborated by a simulation study, where we show that, as the sample size increases, the empirical distribution of the estimation error approaches the asymptotic distribution with accuracy. Furthermore, the simulation study confirms that the Fourier leverage estimator obtained by means of the Fejer kernel leads to a superior finite-sample efficiency in terms of mean squared error.

Finally, we exploit the availability of efficient leverage estimates to investigate the contribution of the leverage effect to the prediction of the future integrated volatility. In [Mykland and Wang, 2014], the authors suggest that adding an extra term, namely the asset return scaled by the leverage effect, to any auto-regressive model aimed at predicting next-period's volatility may increase the model's explanatory power in a statistically significant manner, based on empirical evidence obtained from Microsoft high-frequency prices over the period 2008-2011. Accordingly, in this paper we add the extra term represented by the asset return scaled by the Fourier estimate of the leverage effect to the popular Heterogeneous Auto-Regressive (HAR) volatility model by [Corsi, 2009] and show, using S&P500 prices over the period 2006-2018, that the contribution of this extra term is statistically significant, thus confirming the empirical result by [Mykland and Wang, 2014] on a different model and data set.

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